Supplementary Material

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A Mathematical Proof of Taylor Exponent

Here we show that the Taylor exponent of an independent and identically distributed (i.i.d.) process is 0.5. A proof in a more general form is shown in (Eisler, Bartos, and Kertész, 2007). This is a known mathematical fact, as found previously in (Yule, 1968).

Proposition 1. *The Taylor exponent of a sequence generated by an i.i.d. process is 0.5.*

Proof. Consider i.i.d. random variables $X_1, \ldots, X_i, \ldots, X_N$, where i denotes the location within a text. For a specific word $w_k \in W$, with W being the set of words, let p_k denote the probability of occurrence of word w_k, i.e., $\mathbb{P}(X_i = w_k) = p_k$ (for all i). Naturally, the expectation $\mathbb{E}$ and variance $\mathbb{V}$ of the count of w_k for X_i are the following:

$$\mathbb{E}[X_i] = p_k, \quad (1)$$

$$\mathbb{V}[X_i] = p_k(1-p_k), \quad (2)$$

which only depend on the constant p_k. With window size Δt, $\mu_k = \Delta t \mathbb{E}[X_i]$. Note that $\sigma_k^2 = \Delta t \mathbb{V}[X_i]$, because

$$\begin{aligned} \sigma_k^2 &= \mathbb{V}\left[\sum_{i=1}^{\Delta t} X_i\right] \\ &= \mathbb{E}\left[\left(\sum_{i=1}^{\Delta t}(X_i - p_k)\right)^2\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{\Delta t}(X_i - p_k)^2 \right. \\ &\qquad \left. +2\sum_{i \neq j}(X_i - p_k)(X_j - p_k)\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{\Delta t}(X_i - p_k)^2\right] \\ &= \sum_{i=1}^{\Delta t}\mathbb{V}[X_i] \\ &= \Delta t \mathbb{V}[X_i]. \end{aligned}$$

Furthermore, note that $\mathbb{E}[(X_i - p_k)(X_j - p_k)] = 0$ for every i, j with $i \neq j$, because X_i and X_j are independent of each other and (1) holds. Therefore, Taylor exponent α of an i.i.d. process is 0.5, because

$$\sigma_k^2 = \frac{\mathbb{V}[X_i]}{\mathbb{E}[X_i]}\mu_k.$$

□

References

Eisler, Zoltán, Imre Bartos, and János Kertész. 2007. Fluctuation scaling in complex systems: Taylor's law and beyond. *Advances in Physics*, pages 89–142.

Yule, G. Udny. 1968. *The Statistical Study of Literary Vocabulary*. Archon Books.

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