Embedding Intensional Semantics into Inquisitive Semantics

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Abstract

Ciardelli, Roelofsen, and Theiler (2017) have shown how a Montague-like semantic framework based on inquisitive logic allows for a uniform compositional treatment of both declarative and interrogative constructs. In this setting, a natural question is the one of the relation between the intensional and the inquisitive interpretation of a declarative sentence. We tackle this problem by defining an embedding of intensional semantics into inquisitive semantics, in the spirit of de Groote's and Kanazawa's (2013) intensionalization procedure. We show that the resulting *inquisitivation* procedure preserves intensional validity and entailment.

1 Introduction

Inquisitive semantics (Ciardelli et al., 2013, 2018) provides a semantic framework for analysing the information conveyed by linguistic utterances. It is based on a formal notion of *issue* that is reminiscent of alternative semantics and that allows several linguistic constructs to be assigned a meaning. In particular, it offers a uniform treatment of both declarative and interrogative forms.

Taking advantage of this new semantic framework, Ciardelli, Roelofsen, and Theiler (2017) have introduced a typed inquisitive logic, based on the simply typed λ-calculus, that can be used to provide a compositional semantics to fragments of language that contain interrogative constructs. This opens the door to Montague grammars based on inquisitive logic, but raises the question of the relation between the intensional and the inquisitive interpretation of a declarative utterance. If one sticks to the case of first-order logic, the question can be easily settled. This, unfortunately, is not sufficient. Indeed, a Montague grammar typically contains higher-order constructs, as in the following example that might correspond to the lexical semantic of the word *seek*:

$$\lambda os.\, s\,(\lambda x.\, \mathbf{try}\, x\,(\lambda x.\, o\,(\lambda y.\, \mathbf{find}\, x\, y)))$$

In the above λ-term, constant $\mathtt{try}$ is assigned the following type:

$$\mathbf{e} \to (\mathbf{e} \to \mathbf{s} \to \mathbf{t}) \to \mathbf{s} \to \mathbf{t}$$

Then, the question we must answer is the following one: which inquisitive interpretation should we assign to constant $\mathtt{try}$ so that the intended meaning of *seek* is preserved?

In order to solve this problem, we propose an *inquisitivation* procedure akin to de Groote's and Kanazawa's (2013) intensionalization. This procedure is based on an embedding of the intensional interpretations of the types into their inquisitive interpretations. We then prove that our inquisitivation procedure is adequate in the sense that it preserves validity and entailment.

The rest of our paper is organized as follows:

- Section 2 contains a brief introduction to inquisitive semantics.
- In Section 3, we present the necessary mathematical preliminaries, and we fix the type-theoretic setting in which we are working. In particular, we remind one of the definition of the simply typed λ-calculus, and we give its intensional interpretation.
- In Section 4, we define an embedding of intensional semantics into inquisitive semantics, and we provide the simply typed λ-calculus, with an inquisitive interpretation.
- Section 5, contains the proof that our inquisitivation procedure preserves validity and entailment.
- In Section 6, we compare the inquisitive interpretations of the logical connectives, as defined in

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inquisitive semantics, with their inquisitive interpretations, as resulting from the inquisitivation procedure. We then propose a syntactic translation that allows inquisitive logic to be used as object language.

- In Section 7, we discuss briefly the inquisitivation of modal operators.

2 Inquisitive semantics

Montague semantics, in its original formulation (Montague, 1970, 1973), is only concerned with declarative sentences. To remedy this situation, Hamblin (1973) introduced alternative semantics, which allows for a treatment of interrogative sentences. Hamblin's idea is to interpret a question as the set of its possible answers. Hence, if an answer is modeled by a proposition, a question, in turn, must be modeled by a set of propositions. At the semantic level, a proposition being interpreted as a set s of possible worlds, a question is then interpreted as a set of sets of possible worlds.

This different treatment of declarative vs. interrogative propositions has some disadvantages though. Traditional set-theoretic operations cannot be used on interrogative propositions to interpret common semantic coordination, e.g. conjunction. Moreover, alternatives fail to predict even basic declarative entailments such as *John walks* $\models$ *John moves* (Groenendijk and Stokhof, 1984).

Inquisitive semantics (Ciardelli et al., 2013, 2018) elaborates on the idea of alternative semantics and circumvents some of its drawbacks.

Technically, in inquisitive logic, a *proposition* (also known as an *issue*) is defined to be a non-empty set of sets of possible worlds that is downward-closed with respect to set inclusion. As a consequence, conjunction, disjunction, and entailment can be defined in a standard way, i.e., as intersection, union, and inclusion, respectively. Let us illustrate this by an example.

Consider a discourse universe with three individuals, namely, *Mary*, *John*, and *Ash*, and assume a situation where it is known that exactly one of them is sleeping. Accordingly, we define a set of possible worlds, $W = \{\text{M}, \text{J}, \text{A}\}$, where each possible world corresponds respectively to the fact that Mary, John, or Ash is sleeping. Then, the proposition φ_1 that *Mary sleeps* and the proposition φ_2 that *John sleeps* are interpreted as follows:

$$[\![\varphi_1]\!] = \{\{\text{M}\}, \varnothing\}$$
$$[\![\varphi_2]\!] = \{\{\text{J}\}, \varnothing\}$$

Then, the inquisitive disjunction of φ_1 and φ_2 is interpreted as the union of their interpretations:

$$[\![\varphi_1 \vee \varphi_2]\!] = \{\{\text{M}\}, \{\text{J}\}, \varnothing\}$$

This disjunction does not correspond to a proposition asserting that either Mary or John is sleeping, but rather to the question *whether it is Mary or John who sleeps*. The mere assertion, φ_3, that *Mary or John is sleeping* is interpreted in a different way:

$$[\![\varphi_3]\!] = \{\{\text{M}, \text{J}\}, \{\text{M}\}, \{\text{J}\}, \varnothing\}$$

The proposition, φ_4 asserting that *Mary does not sleep* is interpreted as follows:

$$[\![\varphi_4]\!] = \{\{\text{J}, \text{A}\}, \{\text{J}\}, \{\text{A}\}, \varnothing\}$$

Then, the inquisitive disjunction of φ_1 and φ_4 corresponds to the polar question *whether Mary is sleeping*:

$$[\![\varphi_1 \vee \varphi_4]\!] = \{\{\text{J}, \text{A}\}, \{\text{M}\}, \{\text{J}\}, \{\text{A}\}, \varnothing\}$$

In inquisitive semantics, a proposition has both an informative and an inquisitive content. For instance, the informative content of proposition $\varphi_1 \vee \varphi_2$ is that Ash is not sleeping, and its inquisitive content is the issue whether Mary or John is sleeping. The proposition may then be paraphrased as follows: *knowing that Ash does not sleep, one wonders whether Mary or John is sleeping*. A mere assertion such as φ_1 has a trivial inquisitive content. Its paraphrase would be: *knowing that Mary is sleeping, one wonders whether she is sleeping*. Similarly, a mere question such as $\varphi_1 \vee \varphi_4$ has a trivial informative content: *knowing that Mary sleeps or does not sleep, one wonders whether she is sleeping*. Inquisitive semantics features two projection operators, ! and ?, that respectively trivialize the inquisitive content and the informative content of a proposition. Then, for any proposition φ, one has:

$$\varphi = !\varphi \wedge ?\varphi$$

We end this short introduction to inquisitive semantics by presenting first-order inquisitive logic.

Let $\langle \mathcal{F}, \mathcal{R} \rangle$ be the signature of a first-order language, where $\mathcal{F}$ is the set of function symbols, and

$\mathcal{R}$ is the set of relation symbols. From this signature together with a set $\mathcal{X}$ of first-order variables, the notions of terms and of first-order formulas are defined in the standard way.

The notion of a model does not differ from the one used for intensional logic. A model is a triple $\langle D, W, \mathcal{I}\rangle$, where D is the domain of interpretation, W is the set of possible worlds, an $\mathcal{I}$ is the symbol interpretation function such that:[1]

$$\mathcal{I}(F) \in D^{D^n} \qquad \text{for } F \in \mathcal{F} \text{ of arity } n$$
$$\mathcal{I}(R) \in \mathscr{P}(W)^{D^n} \qquad \text{for } R \in \mathcal{R} \text{ of arity } n$$

Given a valuation ξ from $\mathcal{X}$ into D, the interpretation $[\![t]\!]_\xi$ of a term t is defined as usual, and the interpretation of a first-order formula is given by the following equations:

$$\begin{aligned}
[\![R(t_1,\ldots,t_n)]\!]_\xi &= \mathscr{P}(\mathcal{I}(R)([\![t_1]\!]_\xi,\ldots,[\![t_n]\!]_\xi)) \\
[\![\neg\varphi]\!]_\xi &= \{s \mid \forall t \in [\![\varphi]\!]_\xi.\, s \cap t = \varnothing\} \\
[\![\varphi \wedge \psi]\!]_\xi &= [\![\varphi]\!]_\xi \cap [\![\psi]\!]_\xi \\
[\![\varphi \vee \psi]\!]_\xi &= [\![\varphi]\!]_\xi \cup [\![\psi]\!]_\xi \\
[\![\varphi \to \psi]\!]_\xi &= \\
&\{s \mid \forall t \subseteq s.\, t \in [\![\varphi]\!]_\xi \to t \in [\![\psi]\!]_\xi\} \\
[\![\forall x.\,\varphi]\!]_\xi &= \textstyle\bigcap_{d\in D}[\![\varphi]\!]_{\xi[x:=d]} \\
[\![\exists x.\,\varphi]\!]_\xi &= \textstyle\bigcup_{d\in D}[\![\varphi]\!]_{\xi[x:=d]}
\end{aligned}$$

As for the projection operators ! and ?, they may be added as defined connectives:

$$\begin{aligned}
!\varphi &= \neg\neg\varphi \\
?\varphi &= \varphi \vee \neg\varphi
\end{aligned}$$

3 Type-theoretic setting

Since Montague (1973), it is usual in the field of natural language semantics to use the simply typed λ-calculus as an object language to express the compositional semantics of linguistic constructs. In this paper, we adhere to this tradition, and we take advantage of the present section to remind the reader of some notions related to the simply typed λ-calculus, in order to fix the notations.

We take for granted the notions of (untyped) λ-term, β-redex, β-reduction, and β-equivalence. The notations we use, when they are not explicitly introduced, are taken from (Barendregt, 1984). In particular, we write $t \twoheadrightarrow_\beta u$ for the relation of β-reduction.

A λ-term that does not contain any β-redex is called β-normal (or normal, for short). This notion can be explicitly defined in a syntactic way.

Definition 1. The notions of a neutral λ-term and of a normal λ-term are defined by mutual recursion as follows:

1. every λ-variable is a neutral λ-term;
2. every constant is a neutral λ-term;
3. if t a neutral λ-term and u a normal λ-term, then $t\,u$ is a neutral λ-term.
4. every neutral λ-term is a normal λ-term;
5. if t a normal λ-term, so is $\lambda x.\, t$.

The object language we consider comprises two atomic types: IND (the type of individuals) and PROP (the type of proposition).[2] Accordingly, the definition of a simple type is the following one.

Definition 2. The set of simple types $\mathscr{T}$ is inductively defined as follows:

1. IND, PROP $\in \mathscr{T}$;
2. if $\alpha, \beta \in \mathscr{T}$ then $(\alpha \to \beta) \in \mathscr{T}$.

We provide the λ-terms with a type system *à la Church*. To this end, we consider a pairwise disjoint family of countable sets of λ-variables, $(\mathcal{X}_\alpha)_{\alpha\in\mathscr{T}}$, and a pairwise disjoint family of countable sets of constants, $(\mathcal{C}_\alpha)_{\alpha\in\mathscr{T}}$. Given these two families of sets, the notion of a simply-typed lambda-term obeys the next definition.

Definition 3. The family of sets $(\Lambda_\alpha)_{\alpha\in\mathscr{T}}$ of simply-typed λ-terms of type α is inductively defined as follows:

1. For all $\alpha \in \mathscr{T}$, $\mathcal{X}_\alpha \subseteq \Lambda_\alpha$;
2. For all $\alpha \in \mathscr{T}$, $\mathcal{C}_\alpha \subseteq \Lambda_\alpha$;
3. For all $\alpha, \beta \in \mathscr{T}$, if $t \in \Lambda_{\alpha\to\beta}$ and $u \in \Lambda_\alpha$ then $(t\,u) \in \Lambda_\beta$;
4. For all $\alpha, \beta \in \mathscr{T}$, if $x \in \mathcal{X}_\alpha$ and $t \in \Lambda_\beta$ then $(\lambda x.\, t) \in \Lambda_{\alpha\to\beta}$.

We usually let x range over λ-variables, c over constants, and t, u (possibly with subscripts) over λ-terms. If $t \in \Lambda_\alpha$, we say that the term t is of type α, or that α is the type of t. In order to stress the

[1] For the sake of simplicity, we use rigid models, i.e., models in which the interpretation of a term does not vary from one possible world to the other. This assumption does not affect the results we establish in this paper.

[2] IND and PROP are reminiscent of Montague's $\mathbf{e}$ and $\mathbf{t}$, respectively. It is not the case, however, that PROP will be semantically interpreted as the set $\{0, 1\}$.

type of a term, we sometimes decorate it with types, using an exponent like notation. For instance, we write $(\lambda x^{\alpha}.\, t^{\beta})$ when $(\lambda x.\, t) \in \Lambda_{\alpha \to \beta}$.

The simply-typed λ-terms enjoy several interesting properties, in particular, the subject-reduction property and the normalisation-property. The first one says that the sets $(\Lambda_{\alpha})_{\alpha \in \mathscr{T}}$ are closed by β-reduction. The second one says that every simply-typed λ-term has a normal form. We state them explicitly because we will use them in the sequel.

Proposition 4 (Subject Reduction)**.** *Let t and u be two λ-terms such that $t \twoheadrightarrow_{\beta} u$. If $t \in \Lambda_{\alpha}$ then $u \in \Lambda_{\alpha}$.*

Proposition 5 (Normalization)**.** *Let $t \in \Lambda_{\alpha}$. Then there exists a β-normal form $u \in \Lambda_{\alpha}$ such that $t \twoheadrightarrow_{\beta} u$.*

We end this section by providing the simple types and the simply-typed λ-terms with their set-theoretic semantic interpretation.

In order to give a semantic interpretation to the types, we posit two sets, D and W, that are used to give an interpretation to the atomic types. D, the *domain of interpretation*, is the semantic counterpart of type IND. As for W, the set of *possible worlds*, it is used to provide a semantic interpretation to type PROP.

Definition 6. The semantic interpretation $[\alpha]_{\mathsf{i}}$ of a simple type α is inductively defined by the following equations.

$$[\text{IND}]_{\mathsf{i}} = D$$
$$[\text{PROP}]_{\mathsf{i}} = \mathscr{P}(W)$$
$$[\alpha \to \beta]_{\mathsf{i}} = [\beta]_{\mathsf{i}}{}^{[\alpha]_{\mathsf{i}}}$$

According to the above definition, a type $\alpha \to \beta$ is interpreted in standard[3] way as the set of set-theoretic functions from the interpretation of α into the interpretation of β. The interpretation of type PROP, however, is not the set of Booleans, $\{0, 1\}$, but the powerset of the set of possible worlds. This corresponds to an intensional (or modal) interpretation, where a proposition is interpreted as a subset of the set of possible worlds (hence, the subscript i in the notation).

We now turn to the interpretation of the λ-terms. To this end, we introduce the notion of a model.

[3] We use the so-called *standard* interpretation just to keep the definition of a model simple. In fact, everything we do in this paper could be done in the more general setting of Henkin models (Henkin, 1950).

Definition 7. A model $\mathcal{M} = \langle D, W, \mathcal{I} \rangle$ consists of:

1. a set D, called the domain of interpretation;
2. a set W, called the set of possible worlds;
3. a family $\mathcal{I} = (\mathcal{I}_{\alpha})_{\alpha \in \mathscr{T}}$ of interpretation functions $\mathcal{I}_{\alpha}$ from $\mathcal{C}_{\alpha}$ into $[\alpha]_{\mathsf{i}}$.

From now on and throughout the rest of this paper, we consider that a such a model $\mathcal{M} = \langle D, W, \mathcal{I} \rangle$ is given.

The third component of the model, namely $\mathcal{I}$, allows the constant to be given an interpretation. We need a similar notion in order to interpret the λ-variables. Accordingly, we define a valuation $\xi = (\xi_{\alpha})_{\alpha \in \mathscr{T}}$ to be a family of functions ξ_{α} from $\mathcal{X}_{\alpha}$ into $[\alpha]_{\mathsf{i}}$. Let $\xi = (\xi_{\alpha})_{\alpha \in \mathscr{T}}$ be such a valuation, and let $x \in \mathcal{X}_{\alpha}$ and $a \in [\alpha]_{\mathsf{i}}$. Then, $\xi[x{:=}a]$ stands for the valuation $(\xi'_{\alpha})_{\alpha \in \mathscr{T}}$ such that:

1. $\xi'_{\alpha}(x) = a$;
2. for every $y \in \mathcal{X}_{\alpha}$, if $y \neq x$ then $\xi'_{\alpha}(y) = \xi_{\alpha}(y)$;
3. for every $\beta \in \mathscr{T}$, if $\beta \neq \alpha$ then $\xi'_{\beta} = \xi_{\beta}$.

We are now in a position of defining the interpretation of the λ-terms.

Definition 8. Let $\xi = (\xi_{\alpha})_{\alpha \in \mathscr{T}}$ be a valuation. The interpretation $[\![t]\!]_{\mathsf{i}\,\xi}$ of a λ-term t is inductively defined by the following equations:

$$[\![x^{\alpha}]\!]_{\mathsf{i}\,\xi} = \xi_{\alpha}(x)$$
$$[\![c^{\alpha}]\!]_{\mathsf{i}\,\xi} = \mathcal{I}_{\alpha}(c)$$
$$[\![t^{\alpha \to \beta}\, u^{\alpha}]\!]_{\mathsf{i}\,\xi} = [\![t^{\alpha \to \beta}]\!]_{\mathsf{i}\,\xi}([\![u^{\alpha}]\!]_{\mathsf{i}\,\xi})$$
$$[\![\lambda x^{\alpha}.\, t^{\beta}]\!]_{\mathsf{i}\,\xi} = a \in [\alpha]_{\mathsf{i}} \mapsto [\![t^{\beta}]\!]_{\mathsf{i}\,\xi[x^{\alpha}:=a]}$$

The semantic interpretation of Definition 8 is sound with respect to β-equivalence. We state this proposition explicitly because we will use it later on.

Proposition 9 (Soundness)**.** *Let $\alpha \in \mathscr{T}$, $t, u \in \Lambda_{\alpha}$, and ξ be any valuation. If $t =_{\beta} u$ then $[\![t]\!]_{\mathsf{i}\,\xi} = [\![u]\!]_{\mathsf{i}\,\xi}$.*

The interpretation $[\![t]\!]_{\mathsf{i}\,\xi}$ of a closed λ-term t does not depend upon the valuation ξ. Accordingly, when t is a closed term, we simply write $[\![t]\!]_{\mathsf{i}}$ to denote its interpretation.

A closed λ-term of type PROP is called a formula. Let φ be a formula. We say that φ is valid, and we write $\models_{\mathsf{i}} \varphi$, if and only if $[\![\varphi]\!]_{\mathsf{i}} = W$. Similarly, we say that a sequence of formulas $\varphi_1, \ldots, \varphi_n$ entails a formula φ, which we write $\varphi_1, \ldots, \varphi_n \models_{\mathsf{i}} \varphi$, if and only if $[\![\varphi_1]\!]_{\mathsf{i}} \cap \cdots \cap [\![\varphi_n]\!]_{\mathsf{i}} \subseteq [\![\varphi]\!]_{\mathsf{i}}$.

4 Inquisitivation

Definitions 6 and 8 provide to the types and the terms of the object language an intensional interpretation. As explained in the introduction, our objective is to built from this intensional interpretation an inquisitive one.

A first step towards this goal is to provide the type system with an inquisitive interpretation. This consists mainly in interpreting type PROP as the set of inquisitive propositions, i.e., as the set of *sets of sets of possible worlds*. This motivates the next definition.

Definition 10. The inquisitive semantic interpretation $[\alpha]_\mathsf{i}$ of a simple type α is inductively defined by the following equations.

$$[\text{IND}]_\mathsf{q} = D$$
$$[\text{PROP}]_\mathsf{q} = \mathscr{P}(\mathscr{P}(W))$$
$$[\alpha \to \beta]_\mathsf{q} = [\beta]_\mathsf{q}{}^{[\alpha]_\mathsf{q}}$$

The next step would be to adapt Definition 8 to the inquisitive case. This adaptation seems almost straightforward, except for the constants. Indeed, the interpretation function of the model interprets a constant of type α as an element of $[\alpha]_\mathsf{i}$, not as an element of $[\alpha]_\mathsf{q}$. Consequently, what we need is a way of transforming an element of $[\alpha]_\mathsf{i}$ into an element of $[\alpha]_\mathsf{q}$, while preserving the information it carries.

In other words, what we need for each type α is an embedding $\mathbb{E}_\alpha$ from $[\alpha]_\mathsf{i}$ into $[\alpha]_\mathsf{q}$. At the level of type PROP, such an embedding exists. Indeed, if $A \subseteq W$ is an intensional proposition, $\mathscr{P}(A)$ is an inquisitive proposition that is purely informative and that carries the same informative content as A.

Now, in order to lift up this embedding at every type, we also need projection operators, $\mathbb{P}_\alpha$, from $[\alpha]_\mathsf{q}$ onto $[\alpha]_\mathsf{i}$. Again, at the level of type PROP, such a projection exists. It consists of the operation that takes the union of all the elements of a set of sets. Indeed, for every set A, we have that $\bigcup \mathscr{P}(A) = A$. It remains to lift up this embedding-projection pair at every type. This is achieved by the next definition.

Definition 11. The family of embeddings $(\mathbb{E}_\alpha)_{\alpha \in \mathscr{T}}$ and the family of projections $(\mathbb{P}_\alpha)_{\alpha \in \mathscr{T}}$ are defined by mutual recursion over the types as follows:

$$\mathbb{E}_{\text{IND}}(a) = a$$
$$\mathbb{E}_{\text{PROP}}(p) = \mathscr{P}(p)$$
$$\mathbb{E}_{\alpha \to \beta}(f)(a) = \mathbb{E}_\beta(f(\mathbb{P}_\alpha(a)))$$

$$\mathbb{P}_{\text{IND}}(a) = a$$
$$\mathbb{P}_{\text{PROP}}(p) = \bigcup p$$
$$\mathbb{P}_{\alpha \to \beta}(f)(a) = \mathbb{P}_\beta(f(\mathbb{E}_\alpha(a)))$$

The operators $\mathbb{E}_\alpha$ are what we need to give an inquisitive version of Definition 8. First, let us define an inquisitive valuation to be a family of functions $\xi = (\xi_\alpha)_{\alpha \in \mathscr{T}}$ from $\mathcal{X}_\alpha$ into $[\alpha]_\mathsf{q}$. The inquisitive interpretation of a λ-term is then defined as follows.

Definition 12. Let $\xi = (\xi_\alpha)_{\alpha \in \mathscr{T}}$ be an inquisitive valuation. The inquisitive interpretation $[\![t]\!]_{\mathsf{q}\,\xi}$ of a λ-term t is inductively defined by the following equations:

$$[\![x^\alpha]\!]_{\mathsf{q}\,\xi} = \xi_\alpha(x)$$
$$[\![c^\alpha]\!]_{\mathsf{q}\,\xi} = \mathbb{E}_\alpha(\mathcal{I}_\alpha(c))$$
$$[\![t^{\alpha \to \beta}\, u^\alpha]\!]_{\mathsf{q}\,\xi} = [\![t^{\alpha \to \beta}]\!]_{\mathsf{q}\,\xi}([\![u^\alpha]\!]_{\mathsf{q}\,\xi})$$
$$[\![\lambda x^\alpha .\, t^\beta]\!]_{\mathsf{q}\,\xi} = a \in [\alpha]_\mathsf{q} \mapsto [\![t^\beta]\!]_{\mathsf{q}\,\xi[x:=a]}$$

It turns out that inquisitivation is a particular case of de Groote (2015).[4]

The proof of Proposition 9 does not depend on the interpretation of the constants. Consequently, it also holds for Definition 12.

Proposition 13 (Soundness)**.** *Let $\alpha \in \mathscr{T}$, $t, u \in \Lambda_\alpha$, and ξ be any inquisitive valuation. If $t =_\beta u$ then $[\![t]\!]_{\mathsf{q}\,\xi} = [\![u]\!]_{\mathsf{q}\,\xi}$.*

As for the notions of inquisitive validity and of inquisitive entailment, they are defined as expected: $\models_\mathsf{q} \varphi$ if and only if $[\![\varphi]\!]_\mathsf{q} = \mathscr{P}(W)$, and $\varphi_1, \ldots, \varphi_n \models_\mathsf{q} \varphi$ if and only if $[\![\varphi_1]\!]_\mathsf{q} \cap \cdots \cap [\![\varphi_n]\!]_\mathsf{q} \subseteq [\![\varphi]\!]_\mathsf{q}$.

5 Preservation of validity and entailment

In this section, we prove that our inquisitivation procedure preserves the validity of the propositions, that is, a proposition is valid according to its intensional interpretation, if and only if it is valid according to its inquisitive interpretation. We also establish a similar result for entailment.

[4] In our case, the operators of de Groote (2015) are instantiated so: $T\alpha = \alpha$, $\mathsf{U}\, t = t$, $t \bullet u = t\, u$ and $\mathsf{C}\, t = t$

We start by showing that the operators of embedding and projection are indeed embedding-projection pairs, i.e., that $\mathbb{P}_\alpha \circ \mathbb{E}_\alpha$ is the identity for every type α.

Lemma 14. *Let $\alpha \in \mathscr{T}$ be any type. For all $a \in [\alpha]_{\mathsf{i}}$, $\mathbb{P}_\alpha(\mathbb{E}_\alpha(a)) = a$.*

Proof. The proof proceeds by induction on the structure of α.

1. $\alpha = \text{IND}$.

$$\begin{aligned}\mathbb{P}_{\text{IND}}(\mathbb{E}_{\text{IND}}(a)) &= \mathbb{E}_{\text{IND}}(a)\\ &= a\end{aligned}$$

2. $\alpha = \text{PROP}$.

$$\begin{aligned}\mathbb{P}_{\text{PROP}}(\mathbb{E}_{\text{PROP}}(a)) &= \bigcup \mathbb{E}_{\text{PROP}}(a)\\ &= \bigcup \mathscr{P}(a)\\ &= a\end{aligned}$$

3. $\alpha = \alpha_1 \to \alpha_2$. For all $x \in [\alpha_1]_{\mathsf{i}}$, we have:

$$\begin{aligned}&\mathbb{P}_{\alpha_1\to\alpha_2}(\mathbb{E}_{\alpha_1\to\alpha_2}(a))(x)\\ &\quad= \mathbb{P}_{\alpha_2}(\mathbb{E}_{\alpha_1\to\alpha_2}(a)(\mathbb{E}_{\alpha_1}(x)))\\ &\quad= \mathbb{P}_{\alpha_2}(\mathbb{E}_{\alpha_2}(a(\mathbb{P}_{\alpha_1}(\mathbb{E}_{\alpha_1}(x)))))\\ &\quad= a(\mathbb{P}_{\alpha_1}(\mathbb{E}_{\alpha_1}(x))) \quad \text{by induction hypothesis}\\ &\quad= a(x) \quad \text{by induction hypothesis}\end{aligned}$$

□

It is not the case that $\mathbb{E}_\alpha(\mathbb{P}_\alpha(a)) = a$ for every $a \in [\alpha]_{\mathsf{q}}$. For instance, at type PROP, $\mathbb{E}_{\text{PROP}}(\mathbb{P}_{\text{PROP}}(a)) = \mathscr{P}(\bigcup a)$, which is different from a, in general. In fact, the only inquisitive propositions for which $\mathbb{E}_{\text{PROP}}(\mathbb{P}_{\text{PROP}}(a)) = a$ holds are those propositions that are equal to $\mathscr{P}(b)$ for some $b \subseteq W$. These propositions are called *purely informative* because they do not raise any issue. We generalize this notion by defining an element $a \in [\alpha]_{\mathsf{q}}$ to be *purely informative* if and only if there exists some $b \in [\alpha]_{\mathsf{i}}$ such that $a = \mathbb{E}_\alpha(b)$.

Lemma 15. *Let $\alpha \in \mathscr{T}$ be any type, and let $a \in [\alpha]_{\mathsf{q}}$. $\mathbb{E}_\alpha(\mathbb{P}_\alpha(a)) = a$ if and only if a is purely informative.*

Proof. If $\mathbb{E}_\alpha(\mathbb{P}_\alpha(a)) = a$ then a is purely informative, by definition.

Now suppose that a is purely informative, i.e., that there exists $b \in [\alpha]_{\mathsf{i}}$ such that $a = \mathbb{E}_\alpha(b)$. Then, we have:

$$\begin{aligned}\mathbb{E}_\alpha(\mathbb{P}_\alpha(a)) &= \mathbb{E}_\alpha(\mathbb{P}_\alpha(\mathbb{E}_\alpha(b)))\\ &= \mathbb{E}_\alpha(b) \quad \text{by Lemma 14}\\ &= a\end{aligned}$$

□

We are now in a position of stating and proving the main technical lemma of this section, from which we will derive conservativity results. We first introduce some additional vocabulary and notation.

Let $\xi = (\xi_\alpha)_{\alpha\in\mathscr{T}}$ be an inquisitive valuation. We say that ξ is purely informative if and only if for every $\alpha \in \mathscr{T}$ and $x \in \mathscr{X}_\alpha$, $\xi_\alpha(x)$ is a purely informative element of $[\alpha]_{\mathsf{q}}$.

For $\xi = (\xi_\alpha)_{\alpha\in\mathscr{T}}$ an inquisitive valuation, we write $\mathbb{P} \circ \xi$ for the intensional valuation $(\mathbb{P}_\alpha \circ \xi_\alpha)_{\alpha\in\mathscr{T}}$, i.e., the intensional valuation $\xi' = (\xi'_\alpha)_{\alpha\in\mathscr{T}}$ such that $\xi'_\alpha(x) = \mathbb{P}_\alpha(\xi_\alpha(x))$.

Lemma 16. *Let $t \in \Lambda_\alpha$ be any λ-term of type α, and let ξ be an inquisitive valuation that is purely informative.*

(a) *If t is neutral, $[\![t]\!]_{\mathsf{q}\,\xi} = \mathbb{E}_\alpha([\![t]\!]_{\mathsf{i}\,\mathbb{P}\circ\xi})$.*

(b) *If t is normal, $\mathbb{P}_\alpha([\![t]\!]_{\mathsf{q}\,\xi}) = [\![t]\!]_{\mathsf{i}\,\mathbb{P}\circ\xi}$.*

Proof. We prove both (a) and (b) by a simultaneous induction on the structure of t.

1. $t = x$.
   (a) $$\begin{aligned}[\![x]\!]_{\mathsf{q}\,\xi} &= \xi_\alpha(x)\\ &= \mathbb{E}_\alpha(\mathbb{P}_\alpha(\xi_\alpha(x))) \quad \text{by Lemma 15}\\ &= \mathbb{E}_\alpha([\![x]\!]_{\mathsf{i}\,\mathbb{P}\circ\xi})\end{aligned}$$
   (b) Follows from (a), by Lemma 14.

2. $t = c$.
   (a) $$\begin{aligned}[\![c]\!]_{\mathsf{q}\,\xi} &= \mathbb{E}_\alpha(\mathcal{I}_\alpha(c))\\ &= \mathbb{E}_\alpha([\![c]\!]_{\mathsf{i}\,\mathbb{P}\circ\xi})\end{aligned}$$
   (b) Follows from (a), by Lemma 14.

3. $t = t_1\,t_2$, with $t_1 \in \Lambda_{\beta\to\alpha}$ and $t_2 \in \Lambda_\beta$, for some type β.
   (a) $$\begin{aligned}[\![t_1\,t_2]\!]_{\mathsf{q}\,\xi} &= [\![t_1]\!]_{\mathsf{q}\,\xi}([\![t_2]\!]_{\mathsf{q}\,\xi})\\ &= \mathbb{E}_{\beta\to\alpha}([\![t_1]\!]_{\mathsf{i}\,\mathbb{P}\circ\xi})([\![t_2]\!]_{\mathsf{q}\,\xi})\\ &\qquad \text{by induction hypothesis (a)}\\ &= \mathbb{E}_\alpha([\![t_1]\!]_{\mathsf{i}\,\mathbb{P}\circ\xi}(\mathbb{P}_\beta([\![t_2]\!]_{\mathsf{q}\,\xi})))\\ &= \mathbb{E}_\alpha([\![t_1]\!]_{\mathsf{i}\,\mathbb{P}\circ\xi}([\![t_2]\!]_{\mathsf{i}\,\mathbb{P}\circ\xi}))\\ &\qquad \text{by induction hypothesis (b)}\\ &= \mathbb{E}_\alpha([\![t_1\,t_2]\!]_{\mathsf{i}\,\mathbb{P}\circ\xi})\end{aligned}$$
   (b) Follows from (a), by Lemma 14.

4. $t = \lambda x.\,t_1$, with $x \in \mathscr{X}_{\alpha_1}$ and $t_1 \in \Lambda_{\alpha_2}$, for some types α_1 and α_2.
   (a) Holds vacuously beause t is not neutral.
   (b) For every $a \in [\alpha_1]_{\mathsf{i}}$, we have:

$$
\begin{aligned}
&\mathbb{P}_{\alpha_1\to\alpha_2}([\![\lambda x.\,t_1]\!]_{\mathsf{q}\,\xi})(a)\\
&\quad= \mathbb{P}_{\alpha_2}([\![\lambda x.\,t_1]\!]_{\mathsf{q}\,\xi}(\mathbb{E}_{\alpha_1}\,a))\\
&\quad= \mathbb{P}_{\alpha_2}((a \mapsto [\![t_1]\!]_{\mathsf{q}\,\xi[x:=a]})(\mathbb{E}_{\alpha_1}(a)))\\
&\quad= \mathbb{P}_{\alpha_2}([\![t_1]\!]_{\mathsf{q}\,\xi[x:=\mathbb{E}_{\alpha_1}(a)]})\\
&\quad= [\![t_1]\!]_{\mathsf{i}\,(\mathbb{P}\circ(\xi[x:=\mathbb{E}_{\alpha_1}\,a]))}\\
&\qquad\qquad \text{by induction hypothesis (b)}\\
&\quad= [\![t_1]\!]_{\mathsf{i}\,\mathbb{P}\circ\xi[x:=\mathbb{P}_{\alpha_1}(\mathbb{E}_{\alpha_1}(a))]}\\
&\quad= [\![t_1]\!]_{\mathsf{i}\,\mathbb{P}\circ\xi[x:=a]} \qquad \text{by Lemma 14}\\
&\quad= (a \mapsto [\![t_1]\!]_{\mathsf{i}\,\mathbb{P}\circ\xi[x:=a]})(a)\\
&\quad= [\![\lambda x.\,t_1]\!]_{\mathsf{i}\,\mathbb{P}\circ\xi}(a)
\end{aligned}
$$

□

We may now establish our main result as an immediate consequence of Lemma 16.

Proposition 17. *Let φ be a proposition. Then, $[\![\varphi]\!]_{\mathsf{i}} = a$ if and only if $[\![\varphi]\!]_{\mathsf{q}} = \mathscr{P}(a)$.*

Proof. Suppose $[\![\varphi]\!]_{\mathsf{i}} = a$. Since φ is a simply-typed λ-term, it has a β-normal form φ', which is of type PROP by Proposition 4. Then, φ' being a normal form of atomic type, it is neutral. Hence:

$$
\begin{aligned}
[\![\varphi]\!]_{\mathsf{q}} &= [\![\varphi']\!]_{\mathsf{q}} && \text{by Proposition 13}\\
&= \mathbb{E}_{\text{PROP}}([\![\varphi']\!]_{\mathsf{i}}) && \text{by Lemma 16(a)}\\
&= \mathbb{E}_{\text{PROP}}([\![\varphi]\!]_{\mathsf{i}}) && \text{by Proposition 9}\\
&= \mathbb{E}_{\text{PROP}}(a)\\
&= \mathscr{P}(a)
\end{aligned}
$$

Conversely, if $[\![\varphi]\!]_{\mathsf{q}} = \mathscr{P}(a)$, we obtain the expected result in a similar way, using Lemma 16 (b). □

As a particular case of Proposition 17, we obtain that our inquisitivation procedure preserves validity.

Corollary 18. *Let φ be a proposition. Then, $\models_{\mathsf{q}} \varphi$ if and only if $\models_{\mathsf{i}} \varphi$.*

Finally, observing that $A \subseteq B$ if and only if $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, and that $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$, we obtain that entailment is also preserved.

Corollary 19. *Let $\varphi, \varphi_1, \ldots, \varphi_n$ be propositions. Then, $\varphi_1, \ldots, \varphi_n \models_{\mathsf{q}} \varphi$ if and only if $\varphi_1, \ldots, \varphi_n \models_{\mathsf{i}} \varphi$.*

6 Using inquisitive logic as the object language

Our inquisitivation procedure, as defined by Definitions 11 and 12, leaves the treatment of the logical connectives completely implicit. Somehow, we assumed that the set of constants $\mathcal{C}_{\text{PROP}\to\text{PROP}}$ contains a constant corresponding to negation, that $\mathcal{C}_{\text{PROP}\to\text{PROP}\to\text{PROP}}$ contains constants corresponding to conjunction, disjunction, and implication, and that $\mathcal{C}_{(\text{IND}\to\text{PROP})\to\text{PROP}}$ contains constants corresponding to the quantifiers. In addition, we also assumed family $\mathcal{I} = (\mathcal{I}_\alpha)_{\alpha\in\mathscr{T}}$ of interpretation functions assigns to the logical connectives their standard intensional meaning. That is:

$$
\begin{aligned}
\mathcal{I}_{\text{PROP}\to\text{PROP}}(\neg) &= a \mapsto W \setminus a\\
\mathcal{I}_{\text{PROP}\to\text{PROP}\to\text{PROP}}(\wedge) &= a\,b \mapsto a \cap b\\
\mathcal{I}_{\text{PROP}\to\text{PROP}\to\text{PROP}}(\vee) &= a\,b \mapsto a \cup b\\
\mathcal{I}_{\text{PROP}\to\text{PROP}\to\text{PROP}}(\to) &= a\,b \mapsto (W \setminus a) \cup b\\
\mathcal{I}_{(\text{IND}\to\text{PROP})\to\text{PROP}}(\forall) &= p \mapsto \textstyle\bigcap_{d\in D} p(d)\\
\mathcal{I}_{(\text{IND}\to\text{PROP})\to\text{PROP}}(\exists) &= p \mapsto \textstyle\bigcup_{d\in D} p(d)
\end{aligned}
$$

Then, according to Definition 12, our inquisitive interpretation of the logical connectives is given by $\mathbb{E} \circ \mathcal{I}$, not by their very inquisitive meaning as defined at the end of Section 2. Spelling it out, our inquisitive interpretation of the logical connectives is as follows:

$$
\begin{aligned}
[\![\neg]\!]_{\mathsf{q}} &= a \mapsto \mathscr{P}(W \setminus (\textstyle\bigcup a))\\
[\![\wedge]\!]_{\mathsf{q}} &= a\,b \mapsto \mathscr{P}((\textstyle\bigcup a) \cap (\textstyle\bigcup b))\\
[\![\vee]\!]_{\mathsf{q}} &= a\,b \mapsto \mathscr{P}((\textstyle\bigcup a) \cup (\textstyle\bigcup b))\\
[\![\to]\!]_{\mathsf{q}} &= a\,b \mapsto \mathscr{P}((W \setminus (\textstyle\bigcup a)) \cup (\textstyle\bigcup b))\\
[\![\forall]\!]_{\mathsf{q}} &= p \mapsto \mathscr{P}(\textstyle\bigcap_{d\in D}(\textstyle\bigcup p(d)))\\
[\![\exists]\!]_{\mathsf{q}} &= p \mapsto \mathscr{P}(\textstyle\bigcup_{d\in D}(\textstyle\bigcup p(d)))
\end{aligned}
$$

Let us call the above interpretation of the logical connectives their *weak inquisitive interpretation*. By contrast, let us call the very inquisitive interpretation of the connectives, as given at the end of Section 2, their *strong inquisitive interpretation*. Then, given an inquisitive valuation ξ, let us define the strong inquisitive interpretation $[\![t]\!]_{\mathsf{sq}\,\xi}$ of a λ-term t as in Definition 12, except for the logical connectives that are assigned their strong inquisitive interpretation.

We now wonder whether the weak and the strong interpretations of the connectives coincide. For negation, this is indeed the case.

Lemma 20.

$$[\![\neg]\!]_{\mathsf{sq}} = [\![\neg]\!]_{\mathsf{q}}$$

Proof.

$$
\begin{aligned}
[\![\neg]\!]_{\mathsf{sq}}(a) &= \{s \mid \forall t \in a.\, s \cap t = \varnothing\} \\
&= \{s \mid \forall w \in s.\, \forall t \in a.\, w \notin t\} \\
&= \{s \mid \forall w \in s.\, w \notin \bigcup a\} \\
&\qquad \text{because } a \text{ is downward-closed} \\
&= \mathscr{P}(W \setminus (\bigcup a)) \\
&= [\![\neg]\!]_{\mathsf{q}}(a)
\end{aligned}
$$

□

For the other connectives, the weak and the strong interpretations do not coincide in general. Let us exhibit some counterexamples. Let $W = \{w, v\}$, $D = \{1, 2\}$, and Define a to be $\{\{w\}, \{v\}, \varnothing\}$. For conjunction, we have:

$$[\![\wedge]\!]_{\mathsf{sq}}(a)(a) = a \cap a = a$$

which is different from:

$$
\begin{aligned}
[\![\wedge]\!]_{\mathsf{q}}(a)(a) &= \mathscr{P}((\bigcup a) \cap (\bigcup a)) \\
&= \mathscr{P}(W)
\end{aligned}
$$

For implication, define b to be $\mathscr{P}(W)$. Then, we have:

$$
\begin{aligned}
&[\![\rightarrow]\!]_{\mathsf{sq}}(b)(a) \\
&\quad = \{s \mid \forall t \subseteq s.\, t \in b \rightarrow t \in a\} \\
&\quad = a
\end{aligned}
$$

which is different from:

$$
\begin{aligned}
[\![\rightarrow]\!]_{\mathsf{q}}(b)(a) &= \mathscr{P}((W \setminus (\bigcup b)) \cup (\bigcup a)) \\
&= \mathscr{P}(W)
\end{aligned}
$$

For universal quantification, define p to be $\{(1, a), (2, a)\}$. We then obtain a counterexample similar to the one for conjunction, with $[\![\forall]\!]_{\mathsf{sq}}(p) = a$, which is different from $[\![\forall]\!]_{\mathsf{q}}(p) = \mathscr{P}(W)$.

For disjunction, define c to be $\{\{w\}, \varnothing\}$, and d to be $\{\{v\}, \varnothing\}$. Then we have:

$$
\begin{aligned}
[\![\vee]\!]_{\mathsf{sq}}(c)(d) &= c \cup d \\
&= \{\{w\}, \{v\}, \varnothing\}
\end{aligned}
$$

which is different from

$$
\begin{aligned}
[\![\vee]\!]_{\mathsf{q}}(c)(d) &= \mathscr{P}((\bigcup c) \cup (\bigcup d)) \\
&= \mathscr{P}(W)
\end{aligned}
$$

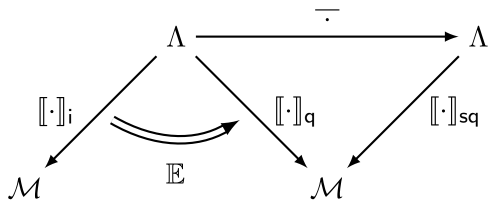
For existential quantification, one obtains a counterexample similar to the one for disjunction by defining q to be $\{(1, c), (2, d)\}$. Then we have that $[\![\exists]\!]_{\mathsf{sq}}(q) = \{\{w\}, \{v\}, \varnothing\}$, and that $[\![\exists]\!]_{\mathsf{q}}(q) = \mathscr{P}(W)$.

Because of the non-coincidence of the weak and the strong inquisitive interpretations of the logical connective, we do not have, in general, that for any formula φ:

$$[\![\varphi]\!]_{\mathsf{q}} = [\![\varphi]\!]_{\mathsf{sq}} \tag{1}$$

Consequently, Proposition 17 in which $[\![\cdot]\!]_{\mathsf{q}}$ would be replaced by $[\![\cdot]\!]_{\mathsf{sq}}$ does not hold.

In order to circumvent this problem, we will introduce a syntactic translation of the λ-terms, $\overline{\cdot}$, such that for every formula φ, $[\![\overline{\varphi}]\!]_{\mathsf{sq}} = [\![\varphi]\!]_{\mathsf{q}}$. With such a translation, the picture of our inquisitivation process is as follows:



Remark, however, that an exact coincidence between the weak and the strong interpretation of the connectives is not needed in order to have that Equation 1 holds. What is needed is that the weak and the strong interpretations coincide on the image of the embedding $\mathbb{E}$, that is that they coincide for the purely informative elements. For conjunction, implication, and universal quantification, this is the case, as shown by the next lemma.

Lemma 21. *Let* $a, b \in [\text{PROP}]_{\mathsf{q}}$, *and* $p \in [\text{IND} \rightarrow \text{PROP}]_{\mathsf{q}}$ *be purely informative elements.*

(a) $[\![\wedge]\!]_{\mathsf{sq}}(a)(b) = [\![\wedge]\!]_{\mathsf{q}}(a)(b)$

(b) $[\![\rightarrow]\!]_{\mathsf{sq}}(a)(b) = [\![\rightarrow]\!]_{\mathsf{q}}(a)(b)$

(c) $[\![\forall]\!]_{\mathsf{sq}}(p) = [\![\forall]\!]_{\mathsf{q}}(p)$

Proof.

(a) Conjunction. Remark that a being purely informative, we have that $a = \mathscr{P}(\bigcup a)$, and similarly for b. Then, we have:

$$
\begin{aligned}
[\![\wedge]\!]_{\mathsf{sq}}(a)(b) &= a \cap b \\
&= \mathscr{P}(\bigcup a) \cap \mathscr{P}(\bigcup b) \\
&= [\![\wedge]\!]_{\mathsf{q}}(a)(b)
\end{aligned}
$$

(b) Implication.

$$\begin{aligned}&[\![\rightarrow]\!]_{\mathsf{sq}}\,(a)(b)\\&=\{s \mid \forall t \subseteq s.\, t \in a \rightarrow t \in b\}\\&=\{s \mid \forall t \subseteq s.\, t \in \mathscr{P}(\bigcup a) \rightarrow t \in \mathscr{P}(\bigcup b)\}\\&=\{s \mid \forall t \subseteq s.\, \forall w \in t.\, w \in (\bigcup a) \rightarrow w \in (\bigcup b)\}\\&=\{s \mid \forall w \in s.\, w \in (W \setminus (\bigcup a)) \cup (\bigcup b)\}\\&=\mathscr{P}((W \setminus (\bigcup a)) \cup (\bigcup b))\\&=[\![\rightarrow]\!]_{\mathsf{q}}\,(a)(b)\end{aligned}$$

(c) Universal quantification. This case is similar to conjunction. □

As for disjunction and existential quantification, their strong and weak interpretations do not coincide, even for the purely informative elements. This is shown, indeed, by the above counterexamples. Nevertheless, we may simulate the weak interpretations of these connective using the projection operator !.

Lemma 22. *Let $\varphi, \psi \in \Lambda_{\text{PROP}}$, $v \in \Lambda_{\text{IND}\rightarrow\text{PROP}}$, and ξ be an inquisitive valuation.*

(a) *If $[\![\varphi]\!]_{\mathsf{q}\,\xi} = [\![\varphi]\!]_{\mathsf{sq}\,\xi}$ and $[\![\psi]\!]_{\mathsf{q}\,\xi} = [\![\psi]\!]_{\mathsf{sq}\,\xi}$ then $[\![\varphi \vee \psi]\!]_{\mathsf{q}\,\xi} = [\![!(\varphi \vee \psi)]\!]_{\mathsf{sq}\,\xi}$.*

(b) *If for all $d \in S$, $[\![v]\!]_{\mathsf{q}\,\xi[x:=d]} = [\![v]\!]_{\mathsf{sq}\,\xi[x:=d]}$ then $[\![\exists x.\, v]\!]_{\mathsf{q}\,\xi} = [\![!(\exists x.\, v)]\!]_{\mathsf{sq}\,\xi}$*

Proof.

(a) Disjunction. Remark that for every inquisitive proposition $a \in \mathscr{P}(\mathscr{P}(W))$, $[\![!]\!]_{\mathsf{sq}}\,(a) = \mathscr{P}(\bigcup a)$.

$$\begin{aligned}[\![\varphi \vee \psi]\!]_{\mathsf{q}\,\xi} &= \mathscr{P}((\bigcup [\![\varphi]\!]_{\mathsf{q}\,\xi}) \cup (\bigcup [\![\psi]\!]_{\mathsf{q}\,\xi}))\\&= \mathscr{P}(\bigcup ([\![\varphi]\!]_{\mathsf{q}\,\xi} \cup [\![\psi]\!]_{\mathsf{q}\,\xi}))\\&= \mathscr{P}(\bigcup ([\![\varphi]\!]_{\mathsf{sq}\,\xi} \cup [\![\psi]\!]_{\mathsf{sq}\,\xi}))\\&= \mathscr{P}(\bigcup [\![\varphi \vee \psi]\!]_{\mathsf{sq}\,\xi})\\&= [\![!(\varphi \vee \psi)]\!]_{\mathsf{sq}\,\xi}\end{aligned}$$

(b) Existential quantification. This case is handled similarly. □

Taking advantage of the above lemma, we define the syntactic translation $\overline{\cdot}$ as follows:

$$\begin{aligned}\overline{x} &= x\\\overline{\neg\varphi} &= \neg\overline{\varphi}\\\overline{\varphi \wedge \psi} &= \overline{\varphi} \wedge \overline{\psi}\\\overline{\varphi \vee \psi} &= !(\overline{\varphi} \vee \overline{\psi})\\\overline{\varphi \rightarrow \psi} &= \overline{\varphi} \rightarrow \overline{\psi}\\\overline{\forall x.\, \varphi} &= \forall x.\, \overline{\varphi}\\\overline{\exists x.\, \varphi} &= !(\exists x.\, \overline{\varphi})\\\overline{c} &= c \qquad \text{for the other constants}\\\overline{t\, u} &= \overline{t}\,\overline{u}\\\overline{\lambda x.\, t} &= \lambda x.\, \overline{t}\end{aligned}$$

Finally, we obtain the following proposition.

Proposition 23. *For any λ-term u, and any inquisitive valuation ξ that is purely informative,*

$$[\![u]\!]_{\mathsf{q}\,\xi} = [\![\overline{u}]\!]_{\mathsf{sq}\,\xi}$$

Proof. By induction over the λ-terms, using Lemmas 20, 21, and 22. □

7 Application to an epistemic modality

Epistemic modalities are logical operators that can be added to intensional logic to model natural language expressions involving the knowledge of an agent. In epistemic logic (Hintikka, 1962), the semantics of *know that + declarative subclause* uses an operator named $\mathbf{K}$.

Ciardelli and Roelofsen (2015) developed a new operator $\mathbf{K}_{\mathsf{q}}$ in view of 1. adapting $\mathbf{K}$ to inquisitive semantics and 2. modeling the semantics of *know + interrogative subclause*.

Let us take the following sentences as illustrations:

(1) a. $\mathbf{Kj}\ (\mathbf{sleep\ m})$ (John knows that Mary sleeps)

b. $\mathbf{Kj}\ (?\,(\mathbf{sleep\ m}))$ (John knows whether Mary sleeps)

c. $\mathbf{Kj}\ (\exists x.\, \mathbf{sleep}\ x)$ (John knows who sleeps)

These last two λ-terms of Λ^{sq} have to be interpreted by the strong inquisitive interpretation so that $\exists$ generates an alternative for every $d \in D$. Similarly, we must interpret $\mathbf{K}$ as the inquisitive epistemic operator of Ciardelli and Roelofsen (2015).

This raises the question whether the strong inquisitive interpretation of (1-a) is still consistent with the one obtained by embedding the intensional version of $\mathbf{K}$.

This section investigates to which group of logical constants $\mathbf{K}$ belongs.

7.1 Traditional modal knowledge

We expose here the traditional treatment of *know* in modal logic (Kripke, 1959).

To interpret $\mathbf{K}$ we need an accessibility relation indexed by individuals $d \in D$:

$$\sigma_d : W \to \mathscr{P}(W)$$

in any model $\mathcal{M}$. In particular, $w \; \sigma_d \; v$ iff agent d cannot distinguish worlds w and v by her knowledge.

Then we can define for every x^{IND} and proposition φ^{PROP},

$$[\![\mathbf{K}\, x\, \varphi]\!]_{\mathsf{i}\,\xi} = \{w \in W \mid \sigma_{[\![x]\!]_{\mathsf{i}\,\xi}}(w) \subseteq [\![\varphi]\!]_{\mathsf{i}\,\xi}\}$$

Embedding this operation yields

$$[\![\mathbf{K}\, x\, \varphi]\!]_{\mathsf{q}\,\xi} = \mathscr{P}(\{w \mid \sigma_{[\![x]\!]_{\mathsf{q}\,\xi}}(w) \subseteq \bigcup [\![\varphi]\!]_{\mathsf{q}\,\xi}\})$$

7.2 Inquisitive knowledge

We can see σ_d as a function from worlds to intensional propositions. The idea of Ciardelli and Roelofsen (2015) is to extend it to a function $\Sigma_d : W \to \mathscr{P}(\mathscr{P}(W))$, mapping worlds to inquisitive propositions, called the inquisitive states of agent d. This way, the inquisitive knowledge modality can take inquisitive propositions as inputs.

The intensional counterpart of Σ_d can be retrieved by taking the truth set of the inquisitive state at world w:

$$\sigma_d(w) = \bigcup(\Sigma_d(w))$$

$\Sigma_d(w)$ represents the issue $\mathcal{P}$ that agent d entertains at world w. The informational content of $\mathcal{P}$ is where d locates the current world, so what d knows. The inquisitive content of $\mathcal{P}$ is related to what d wonders. Therefore, to interpret knowledge, we only need to use $\bigcup \Sigma_d(w)$, i.e. $\sigma_d(w)$.

The strong inquisitive interpretation of the knowledge operator is

$$[\![\mathbf{K}\, x\, \varphi]\!]_{\mathsf{sq}\,\xi} = \{s \mid \forall w \in s.\, \sigma_{[\![x]\!]_{\mathsf{sq}\,\xi}}(w) \in [\![\varphi]\!]_{\mathsf{sq}\,\xi}\}$$

For $\mathbf{K}\, x\, \varphi$ to be true at s, the knowledge of agent x at every world w of s has to settle the proposition expressed by φ. This way, $\mathbf{K}$ can be used to interpret both *know that* + *declarative* and *know* + *interrogative* in a single formulation.

7.3 Inquisitivation of K

The modality $\mathbf{K}$ belongs to group 2: $[\![\mathbf{K}]\!]_{\mathsf{sq}\,\xi}$ coincides with $[\![\mathbf{K}]\!]_{\mathsf{q}\,\xi})$ on the image of $\mathbb{E}$.

Let us first provide a counterexample against their coicidence in the general case.

Again, take the model having $W = \{w, v\}$, $D = \{d\}$ and $\Sigma_d(w) = \Sigma_d(v) = \{\{w\}, \{v\}, \varnothing\}$. Therefore, $\sigma_d(w) = \sigma_d(v) = W$. Set $Q = \Sigma_d(w)$. Then,

$$[\![\mathbf{K}]\!]_{\mathsf{sq}\,\xi}(d)(\mathcal{Q}) = \mathscr{P}(\{w \mid \sigma_d(w) \in \mathcal{Q}\}) = \{\varnothing\}$$

whereas

$$[\![\mathbf{K}]\!]_{\mathsf{q}\,\xi}(d)(\mathcal{Q}) = \mathscr{P}(\{w \mid \sigma_d(w) \subseteq \bigcup \mathcal{Q}\}) = \mathscr{P}(W)$$

Proposition 24. *Let $\mathcal{P}$ be a purely informative issue and $d \in D$.*

$$[\![\mathbf{K}]\!]_{\mathsf{sq}\,\xi}(d)(\mathcal{P}) = [\![\mathbf{K}]\!]_{\mathsf{q}\,\xi}(d)(\mathcal{P})$$

Proof. The derivation goes like this

$$\begin{aligned} [\![\mathbf{K}]\!]_{\mathsf{sq}\,\xi}(d)(\mathcal{P}) &= \{s \mid \forall w \in s.\, \sigma_d(w) \in \mathscr{P}(\bigcup \mathcal{P})\} \\ &= \{s \mid \forall w \in s.\, \sigma_d(w) \subseteq \bigcup \mathcal{P}\} \\ &= \mathscr{P}(\{w \mid \sigma_d(w) \subseteq \bigcup \mathcal{P}\}) \\ &= [\![\mathbf{K}]\!]_{\mathsf{q}\,\xi}(d)(\mathcal{P}) \end{aligned}$$

□

This proves that the inquisitive epistemic modality is indeed a "natural" extension of traditional $\mathbf{K}$, as suggested in Ciardelli and Roelofsen (2015).

8 Conclusion

We designed a transformation that creates inquisitive lexical representations out of intensional lexical interpretations. This transformation, called inquisitivation, can be used as a procedure to embed an intensional interpretation into the inquisitive world, where more operations are available, e.g. to express questions.

We proved that inquisitivation preserves validity and entailment.

We classified logical connectives into three groups w.r.t. how their inquisitivation coincides with their counterparts defined by inquisitive logic. Group 1 includes negation and exhibits an exact coincidence. In group 2, connectives (e.g. conjunction) exhibit a coincidence on the image of inquisitivation (i.e. on purely informative issues). The connectives of group 3 (e.g. disjunction) do not coincide in general. But they are definable by their inquisitive counterpart.

We finally showed that the knowledge operator **K** shares properties with its adaptation to inquisitive logic defined by Ciardelli and Roelofsen (2015). As such, it belongs to group 2.

Inquisitivation offers a tool to easily transfer any system based on intensional semantics to inquisitive semantics. Future works may focus on other such systems, like dynamic semantics.

It would also be interesting to try to emulate an inquisitive logic out of another basis than intensional semantics. For example, events may have a rich enough structure to allow an inquisitive logic based on (set of) events instead of information states.

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