MIX Is Not a Tree-Adjoining Language

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Abstract

The language MIX consists of all strings over the three-letter alphabet $\{a, b, c\}$ that contain an equal number of occurrences of each letter. We prove Joshi's (1985) conjecture that MIX is not a tree-adjoining language.

1 Introduction

The language

$$\text{MIX} = \{\, w \in \{a, b, c\}^* \mid |w|_a = |w|_b = |w|_c \,\}$$

has attracted considerable attention in computational linguistics.[1] This language was used by Bach (1981) in an exercise to show that the permutation closure of a context-free language is not necessarily context-free.[2] MIX may be considered a prototypical example of *free word order language*, but, as remarked by Bach (1981), it seems that no human language "has such complete freedom for order", because "typically, certain constituents act as 'boundary domains' for scrambling". Joshi (1985) refers to MIX as representing "an extreme case of the degree of free word order permitted in a language", which is "linguistically not relevant". Gazdar (1988) adopts a similar position regarding the relation between MIX and natural languages, noting that "it seems rather unlikely that any natural language will turn out to have a MIX-like characteristic".

It therefore seems natural to assume that languages such as MIX should be excluded from any class of formal languages that purports to be a tight formal characterization of the possible natural languages. It was in this spirit that Joshi et al. (1991) suggested that MIX should not be in the class of so-called *mildly context-sensitive* languages:

> "[mildly context-sensitive grammars] capture only certain kinds of dependencies, e.g., nested dependencies and certain limited kinds of cross-serial dependencies (for example, in the subordinate clause constructions in Dutch or some variations of them, but perhaps not in the so-called MIX (or Bach) language) ...."

Mild context-sensitivity is an informally defined notion first introduced by Joshi (1985); it consists of the three conditions of *limited cross-serial dependencies*, *constant growth*, and *polynomial parsing*. The first condition is only vaguely formulated, but the other two conditions are clearly satisfied by tree-adjoining grammars. The suggestion of Joshi et al. (1991) was that MIX should be regarded as a violation of the condition of limited cross-serial dependencies.

Joshi (1985) conjectured rather strongly that MIX is not a tree-adjoining language: "TAGs cannot generate this language, although for TAGs the proof is not in hand yet". An even stronger conjecture was made by Marsh (1985), namely, that MIX is not an

[1]If w is a string and d is a symbol, we write $|w|_d$ to mean the number of occurrences of d in w. We will use the notation $|w|$ to denote the length of w, i.e., the total number of occurrences of symbols in w.

[2]According to Gazdar (1988), "MIX was originally described by Emmon Bach and was so-dubbed by students in the 1983 Hampshire College Summer Studies in Mathematics". According to Bach (1988), the name MIX was "the happy invention of Bill Marsh".

indexed language.[3] (It is known that the indexed languages properly include the tree-adjoining languages.) Joshi et al. (1991), however, expressed a more pessimistic view about the conjecture:

> "It is not known whether TAG ... can generate MIX. This has turned out to be a very difficult problem. In fact, it is not even known whether an IG [(indexed grammar)] can generate MIX."

This open question has become all the more pressing after a recent result by Salvati (2011). This result says that MIX is in the class of *multiple context-free languages* (Seki et al., 1991), or equivalently, languages of *linear context-free rewriting systems* (Vijay-Shanker et al., 1987; Weir, 1988), which has been customarily regarded as a formal counterpart of the informal notion of a mildly context-sensitive language.[4] It means that either we have to abandon the identification of multiple context-free languages with mildly context-sensitive languages, or we should revise our conception of limited cross-serial dependencies and stop regarding MIX-like languages as violations of this condition. Surely, the resolution of Joshi's (1985) conjecture should crucially affect the choice between these two alternatives.

In this paper, we prove that MIX is not a tree-adjoining language. Our proof is cast in terms of the formalism of *head grammar* (Pollard, 1984; Roach, 1987), which is known to be equivalent to TAG (Vijay-Shanker and Weir, 1994). The key to our proof is the notion of an *n-decomposition* of a string over $\{a, b, c\}$, which is similar to the notion of a derivation in head grammars, but independent of any particular grammar. The parameter n indicates how unbalanced the occurrence counts of the three letters can be at any point in a decomposition. We first show that if MIX is generated by some head grammar, then there is an n such that every string in MIX has an n-decomposition. We then prove that if every string in MIX has an n-decomposition, then every string in MIX must have a 2-decomposition. Finally, we exhibit a particular string in MIX that has no 2-decomposition. The length of this string is 87, and the fact that it has no 2-decomposition was first verified by a computer program accompanying this paper. We include here a rigorous, mathematical proof of this fact not relying on the computer verification.

[3]The relation of MIX with indexed languages is also of interest in combinatorial group theory. Gilman (2005) remarks that "it does not ... seem to be known whether or not the word problem of $Z \times Z$ is indexed", alluding to the language $O_2 = \{ w \in \{a, \bar{a}, b, \bar{b}\}^* \mid |w|_a = |w|_{\bar{a}}, |w|_b = |w|_{\bar{b}} \}$. Since O_2 and MIX are rationally equivalent, O_2 is indexed if and only if MIX is indexed (Salvati, 2011).

[4]Joshi et al. (1991) presented linear context-free rewriting systems as mildly context-sensitive grammars. Groenink (1997) wrote "The class of mildly context-sensitive languages seems to be most adequately approached by LCFRS."

2 Head Grammars

A *head grammar* is a quadruple $G = (N, \Sigma, P, S)$, where N is a finite set of nonterminals, Σ is a finite set of terminal symbols (alphabet), S is a distinguished element of N, and P is a finite set of rules. Each nonterminal is interpreted as a binary predicate on strings in Σ^*. There are four types of rules:

$$\begin{aligned}
&A(\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{y}_1, \boldsymbol{y}_2) \leftarrow B(\boldsymbol{x}_1, \boldsymbol{x}_2), C(\boldsymbol{y}_1, \boldsymbol{y}_2)\\
&A(\boldsymbol{x}_1, \boldsymbol{x}_2\boldsymbol{y}_1\boldsymbol{y}_2) \leftarrow B(\boldsymbol{x}_1, \boldsymbol{x}_2), C(\boldsymbol{y}_1, \boldsymbol{y}_2)\\
&A(\boldsymbol{x}_1\boldsymbol{y}_1, \boldsymbol{y}_2\boldsymbol{x}_2) \leftarrow B(\boldsymbol{x}_1, \boldsymbol{x}_2), C(\boldsymbol{y}_1, \boldsymbol{y}_2)\\
&A(w_1, w_2) \leftarrow
\end{aligned}$$

Here, $A, B, C \in N$, $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2$ are *variables*, and $w_1, w_2 \in \Sigma \cup \{\varepsilon\}$.[5] Rules of the first three types are *binary rules* and rules of the last type are *terminating rules*. This definition of a head grammar actually corresponds to a normal form for head grammars that appears in section 3.3 of Vijay-Shanker and Weir's (1994) paper.[6]

The rules of head grammars are interpreted as implications from right to left, where variables can be instantiated to any terminal strings. Each binary

[5]We use ε to denote the empty string.

[6]This normal form is also mentioned in chapter 5, section 4 of Kracht's (2003) book. The notation we use to express rules of head grammars is borrowed from *elementary formal systems* (Smullyan, 1961; Arikawa et al., 1992), also known as *literal movement grammars* (Groenink, 1997; Kracht, 2003), which are logic programs over strings. In Vijay-Shanker and Weir's (1994) notation, the four rules are expressed as follows:

$$\begin{aligned}
&A \to C_{2,2}(B, C)\\
&A \to C_{1,2}(B, C)\\
&A \to W(B, C)\\
&A \to C_{1,1}(w_1 \uparrow w_2)
\end{aligned}$$

rule involves an operation that combines two pairs of strings to form a new pair. The operation involved in the third rule is known as *wrapping*; the operations involved in the first two rules we call *left concatenation* and *right concatenation*, respectively. If $G = (N, \Sigma, P, S)$ is a head grammar, $A \in N$, and $w_1, w_2 \in \Sigma^*$, then we say that a *fact* $A(w_1, w_2)$ is *derivable* and write $\vdash_G A(w_1, w_2)$, if $A(w_1, w_2)$ can be inferred using the rules in P. More formally, we have $\vdash_G A(w_1, w_2)$ if one of the following conditions holds:

- $A(w_1, w_2) \leftarrow$ is a terminating rule in P.
- $\vdash_G B(u_1, u_2)$, $\vdash_G C(v_1, v_2)$, and there is a binary rule $A(\alpha_1, \alpha_2) \leftarrow B(\boldsymbol{x}_1, \boldsymbol{x}_2), C(\boldsymbol{y}_1, \boldsymbol{y}_2)$ in P such that (w_1, w_2) is the result of substituting u_1, u_2, v_1, v_2 for $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2$, respectively, in (α_1, α_2).

The language of G is

$$L(G) = \{\, w_1 w_2 \mid \vdash_G S(w_1, w_2) \,\}.$$

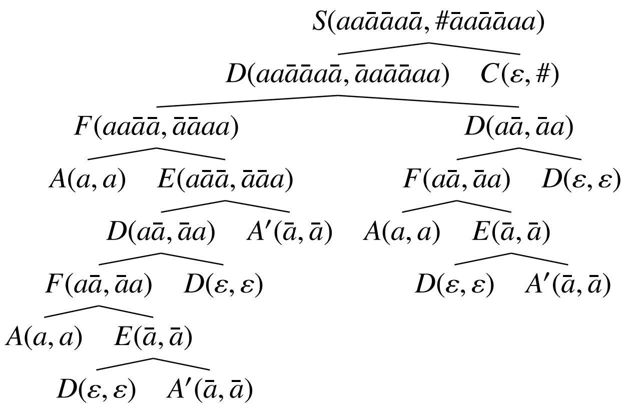
Example 1. Let $G = (N, \Sigma, P, S)$, where $N = \{S, A, A', C, D, E, F\}$, $\Sigma = \{a, \bar{a}, \#\}$, and P consists of the following rules:

$$\begin{aligned}
&S(\boldsymbol{x}_1\boldsymbol{y}_1, \boldsymbol{y}_2\boldsymbol{x}_2) \leftarrow D(\boldsymbol{x}_1, \boldsymbol{x}_2), C(\boldsymbol{y}_1, \boldsymbol{y}_2)\\
&C(\varepsilon, \#) \leftarrow\\
&D(\varepsilon, \varepsilon) \leftarrow\\
&D(\boldsymbol{x}_1\boldsymbol{y}_1, \boldsymbol{y}_2\boldsymbol{x}_2) \leftarrow F(\boldsymbol{x}_1, \boldsymbol{x}_2), D(\boldsymbol{y}_1, \boldsymbol{y}_2)\\
&F(\boldsymbol{x}_1\boldsymbol{y}_1, \boldsymbol{y}_2\boldsymbol{x}_2) \leftarrow A(\boldsymbol{x}_1, \boldsymbol{x}_2), E(\boldsymbol{y}_1, \boldsymbol{y}_2)\\
&A(a, a) \leftarrow\\
&E(\boldsymbol{x}_1\boldsymbol{y}_1, \boldsymbol{y}_2\boldsymbol{x}_2) \leftarrow D(\boldsymbol{x}_1, \boldsymbol{x}_2), A'(\boldsymbol{y}_1, \boldsymbol{y}_2)\\
&A'(\bar{a}, \bar{a}) \leftarrow
\end{aligned}$$

We have $L(G) = \{\, w\#w^R \mid w \in D_{\{a,\bar{a}\}} \,\}$, where $D_{\{a,\bar{a}\}}$ is the Dyck language over $\{a, \bar{a}\}$ and w^R is the reversal of w. All binary rules of this grammar are wrapping rules.

If $\vdash_G A(w_1, w_2)$, a *derivation tree* for $A(w_1, w_2)$ is a finite binary tree whose nodes are labeled by facts that are derived during the derivation of $A(w_1, w_2)$. A derivation tree for $A(w_1, w_2)$ represents a "proof" of $\vdash_G A(w_1, w_2)$, and is formally defined as follows:

- If $A(w_1, w_2) \leftarrow$ is a terminating rule, then a tree with a single node labeled by $A(w_1, w_2)$ is a derivation tree for $A(w_1, w_2)$.



$S(aa\bar{a}\bar{a}a\bar{a}, \#\bar{a}a\bar{a}\bar{a}aa)$
$D(aa\bar{a}\bar{a}a\bar{a}, \bar{a}a\bar{a}\bar{a}aa)$
$C(\varepsilon, \#)$
$F(aa\bar{a}\bar{a}, \bar{a}\bar{a}aa)$
$D(a\bar{a}, \bar{a}a)$
$A(a, a)$
$E(a\bar{a}\bar{a}, \bar{a}\bar{a}a)$
$F(a\bar{a}, \bar{a}a)$
$D(\varepsilon, \varepsilon)$
$D(a\bar{a}, \bar{a}a)$
$A'(\bar{a}, \bar{a})$
$A(a, a)$
$E(\bar{a}, \bar{a})$
$F(a\bar{a}, \bar{a}a)$
$D(\varepsilon, \varepsilon)$
$D(\varepsilon, \varepsilon)$
$A'(\bar{a}, \bar{a})$
$A(a, a)$
$E(\bar{a}, \bar{a})$
$D(\varepsilon, \varepsilon)$
$A'(\bar{a}, \bar{a})$

Figure 1: An example of a derivation tree of a head grammar.

- If $\vdash_G A(w_1, w_2)$ is derived from $\vdash_G B(u_1, u_2)$ and $\vdash_G C(v_1, v_2)$ by some binary rule, then a binary tree whose root is labeled by $A(w_1, w_2)$ and whose immediate left (right) subtree is a derivation tree for $B(u_1, u_2)$ (for $C(v_1, v_2)$, respectively) is a derivation tree for $A(w_1, w_2)$.

If $w \in L(G)$, a derivation tree for w is a derivation tree for some $S(w_1, w_2)$ such that $w_1 w_2 = w$.

Example 1 (continued)**.** Figure 1 shows a derivation tree for $aa\bar{a}\bar{a}a\bar{a}\#\bar{a}a\bar{a}\bar{a}aa$.

The following lemma should be intuitively clear from the definition of a derivation tree:

Lemma 1. *Let $G = (N, \Sigma, P, S)$ be a head grammar and A be a nonterminal in N. Suppose that $w \in L(G)$ has a derivation tree in which a fact $A(v_1, v_2)$ appears as a label of a node. Then there are strings z_0, z_1, z_2 with the following properties:*

(i) $w = z_0 v_1 z_1 v_2 z_2$, *and*

(ii) $\vdash_G A(u_1, u_2)$ *implies* $z_0 u_1 z_1 u_2 z_2 \in L(G)$.

Proof. We can prove by straightforward induction on the height of derivation trees that whenever $A(v_1, v_2)$ appears on a node in a derivation tree for $B(w_1, w_2)$, then there exist z_0, z_1, z_2, z_3 that satisfy one of the following conditions:

(a) $w_1 = z_0 v_1 z_1 v_2 z_2$, $w_2 = z_3$, and $\vdash_G A(u_1, u_2)$ implies $\vdash_G B(z_0 u_1 z_1 u_2 z_2, z_3)$.

(b) $w_1 = z_0$, $w_2 = z_1 v_1 z_2 v_2 z_3$, and $\vdash_G A(u_1, u_2)$ implies $\vdash_G B(z_0, z_1 u_1 z_2 u_2 z_3)$.

(c) $w_1 = z_0 v_1 z_1$, $w_2 = z_2 v_2 z_3$, and $\vdash_G A(u_1, u_2)$ implies $\vdash_G B(z_0 u_1 z_1, z_2 u_2 z_3)$.

We omit the details. □

We call a nonterminal A of a head grammar G *useless* if A does not appear in any derivation trees for strings in $L(G)$. Clearly, useless nonterminals can be eliminated from any head grammar without affecting the language of the grammar.

3 Decompositions of Strings in MIX

Henceforth, $\Sigma = \{a, b, c\}$. Let $\mathbb{Z}$ denote the set of integers. Define functions $\psi_1, \psi_2 \colon \Sigma^* \to \mathbb{Z}$, $\psi \colon \Sigma^* \to \mathbb{Z} \times \mathbb{Z}$ by

$$\begin{aligned} \psi_1(w) &= |w|_a - |w|_c, \\ \psi_2(w) &= |w|_b - |w|_c, \\ \psi(w) &= (\psi_1(w), \psi_2(w)). \end{aligned}$$

Clearly, we have $\psi(a) = (1, 0), \psi(b) = (0, 1), \psi(c) = (-1, -1)$, and

$$w \in \text{MIX} \quad \text{iff} \quad \psi(w) = (0, 0).$$

Note that for all strings $w_1, w_2 \in \Sigma^*$, $\psi(w_1 w_2) = \psi(w_1) + \psi(w_2)$. In other words, ψ is a homomorphism from the free monoid Σ^* to $\mathbb{Z} \times \mathbb{Z}$ with addition as the monoid operation and $(0, 0)$ as identity.

Lemma 2. *Suppose that $G = (N, \Sigma, P, S)$ is a head grammar without useless nonterminals such that $L(G) \subseteq$ MIX. There exists a function $\Psi_G \colon N \to \mathbb{Z} \times \mathbb{Z}$ such that $\vdash_G A(u_1, u_2)$ implies $\psi(u_1 u_2) = \Psi_G(A)$.*

Proof. Since G has no useless nonterminals, for each nonterminal A of G, there is a derivation tree for some string in $L(G)$ in which A appears in a node label. By Lemma 1, there are strings z_0, z_1, z_2 such that $\vdash_G A(u_1, u_2)$ implies $z_0 u_1 z_1 u_2 z_2 \in L(G)$. Since $L(G) \subseteq$ MIX, we have $\psi(z_0 u_1 z_1 u_2 z_2) = (0, 0)$, and hence

$$\psi(u_1 u_2) = -\psi(z_0 z_1 z_2). \qquad □$$

A *decomposition* of $w \in \Sigma^*$ is a finite binary tree satisfying the following conditions:

- the root is labeled by some (w_1, w_2) such that $w = w_1 w_2$,
- each internal node whose left and right children are labeled by (u_1, u_2) and (v_1, v_2), respectively, is labeled by one of $(u_1 u_2 v_1, v_2)$, $(u_1, u_2 v_1 v_2)$, $(u_1 v_1, v_2 u_2)$.
- each leaf node is labeled by some (s_1, s_2) such that $s_1 s_2 \in \{b, c\}^* \cup \{a, c\}^* \cup \{a, b\}^*$.

Thus, the label of an internal node in a decomposition is obtained from the labels of its children by left concatenation, right concatenation, or wrapping. It is easy to see that if G is a head grammar over the alphabet Σ, any derivation for $w \in L(G)$ induces a decomposition of w. (Just strip off nonterminals.) Note that unlike with derivation trees, we have placed no bound on the length of a string that may appear on a leaf node of a decomposition. This will be convenient in some of the proofs below.

When p and q are integers, we write $[p, q]$ for the set $\{ r \in \mathbb{Z} \mid p \leq r \leq q \}$. We call a decomposition of w an *n-decomposition* if each of its nodes is labeled by some (v_1, v_2) such that $\psi(v_1 v_2) \in [-n, n] \times [-n, n]$.

Lemma 3. *If MIX = $L(G)$ for some head grammar $G = (\Sigma, N, P, S)$, then there exists an n such that each $w \in$ MIX has an n-decomposition.*

Proof. We may suppose without loss of generality that G has no useless nonterminal. Since MIX = $L(G)$, there is a function Ψ_G satisfying the condition of Lemma 2. Since the set N of nonterminals of G is finite, there is an n such that $\Psi_G(A) \in [-n, n] \times [-n, n]$ for all $A \in N$. Then it is clear that a derivation tree for $w \in L(G)$ induces an n-decomposition of w. □

If $w = d_1 \dots d_m \in \Sigma^m$, then for $0 \leq i \leq j \leq m$, we write $w[i, j]$ to refer to the substring $d_{i+1} \dots d_j$ of w. (As a special case, we have $w[i, i] = \varepsilon$.) The following is a key lemma in our proof:

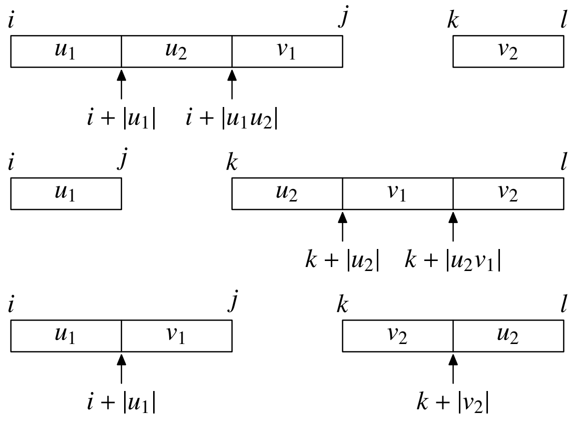
Lemma 4. *If each $w \in$ MIX has an n-decomposition, then each $w \in$ MIX has a 2-decomposition.*

Proof. Assume that each $w \in$ MIX has an n-decomposition. Define a homomorphism $\gamma_n \colon \Sigma^* \to \Sigma^*$ by

$$\begin{aligned} \gamma_n(a) &= a^n, \\ \gamma_n(b) &= b^n, \\ \gamma_n(c) &= c^n. \end{aligned}$$

Clearly, γ_n is an injection, and we have $\psi(\gamma_n(v)) = n \cdot \psi(v)$ for all $v \in \Sigma^*$.

Let $w \in \mathrm{MIX}$ with $|w| = m$. Then $w' = \gamma_n(w) \in \mathrm{MIX}$ and $|w'| = mn$. By assumption, w' has an n-decomposition $\mathcal{D}$. We assign a 4-tuple (i, j, k, l) of natural numbers to each node of $\mathcal{D}$ in such a way that $(w'[i, j], w'[k, l])$ equals the label of the node. This is done recursively in an obvious way, starting from the root. If the root is labeled by (w_1, w_2), then it is assigned $(0, |w_1|, |w_1|, |w_1 w_2|)$. If a node is assigned a tuple (i, j, k, l) and has two children labeled by (u_1, u_2) and (v_1, v_2), respectively, then the 4-tuples assigned to the children are determined according to how (u_1, u_2) and (v_1, v_2) are combined at the parent node:



Now define a function $f\colon [0, mn] \to \{\, kn \mid 0 \le k \le m \,\}$ by

$$f(i) = \begin{cases} i & \text{if } n \text{ divides } i, \\ n \cdot \lfloor i/n \rfloor & \text{if } n \text{ does not divide } i \text{ and} \\ & \qquad w'[i-1, i] \in \{a, b\}, \\ n \cdot \lceil i/n \rceil & \text{if } n \text{ does not divide } i \text{ and} \\ & \qquad w'[i-1, i] = c. \end{cases}$$

Clearly, f is weakly increasing in the sense that $i \le j$ implies $f(i) \le f(j)$. Let $\mathcal{D}'$ be the result of replacing the label of each node in $\mathcal{D}$ by

$$(w'[f(i), f(j)], w'[f(k), f(l)]),$$

where (i, j, k, l) is the 4-tuple of natural numbers assigned to that node by the above procedure. It is easy to see that $\mathcal{D}'$ is another decomposition of w'. Note that since each of $f(i), f(j), f(k), f(l)$ is an integral multiple of n, we always have

$$(w'[f(i), f(j)], w'[f(k), f(l)]) = (\gamma_n(u), \gamma_n(v))$$

for some substrings u, v of w. This implies that for $h = 1, 2$,

$$\psi_h(w'[f(i), f(j)]w'[f(k), f(l)])$$

is an integral multiple of n.

Claim. $\mathcal{D}'$ is a $2n$-decomposition.

We have to show that every node label (v_1, v_2) in $\mathcal{D}'$ satisfies $\psi(v_1 v_2) \in [-2n, 2n] \times [-2n, 2n]$. For $h = 1, 2$, define $\varphi_h\colon [0, mn] \times [0, mn] \to \mathbb{Z}$ as follows:

$$\varphi_h(i, j) = \begin{cases} \psi_h(w'[i, j]) & \text{if } i \le j, \\ -\psi_h(w'[j, i]) & \text{otherwise.} \end{cases}$$

Then it is easy to see that for all $i, j, i', j' \in [0, mn]$,

$$\varphi_h(i', j') = \varphi_h(i', i) + \varphi_h(i, j) + \varphi_h(j, j').$$

Inspecting the definition of the function f, we can check that

$$\varphi_h(f(i), i) \in [0, n-1]$$

always holds. Suppose that (i, j, k, l) is assigned to a node in $\mathcal{D}$. By assumption, we have $\psi_h(w'[i, j]w'[k, l]) \in [-n, n]$, and

$$\begin{aligned}
&\psi_h(w'[f(i), f(j)]w'[f(k), f(l)]) \\
&\quad = \psi_h(w'[f(i), f(j)]) + \psi_h(w'[f(k), f(l)]) \\
&\quad = \varphi_h(f(i), f(j)) + \varphi_h(f(k), f(l)) \\
&\quad = \varphi_h(f(i), i) + \varphi_h(i, j) + \varphi_h(j, f(j)) \\
&\qquad\quad + \varphi_h(f(k), k) + \varphi_h(k, l) + \varphi_h(l, f(l)) \\
&\quad = \varphi_h(f(i), i) + \psi_h(w'[i, j]) + \varphi_h(j, f(j)) \\
&\qquad\quad + \varphi_h(f(k), k) + \psi_h(w'[k, l]) + \varphi_h(l, f(l)) \\
&\quad = \psi_h(w'[i, j]w'[k, l]) + \varphi_h(f(i), i) + \varphi_h(f(k), k) \\
&\qquad\quad + \varphi_h(j, f(j)) + \varphi_h(l, f(l)) \\
&\quad \in \{\, p + q_1 + q_2 + r_1 + r_2 \mid p \in [-n, n], \\
&\qquad\quad q_1, q_2 \in [0, n-1], r_1, r_2 \in [-n+1, 0] \,\} \\
&\quad = [-3n + 2, 3n - 2].
\end{aligned}$$

Since $\psi_h(w'[f(i), f(j)]w'[f(k), f(l)])$ must be an integral multiple of n, it follows that

$$\psi_h(w'[f(i), f(j)]w'[f(k), f(l)]) \in \{-2n, -n, 0, n, 2n\}.$$

This establishes the claim.

We have shown that each node of $\mathcal{D}'$ is labeled by a pair of strings of the form $(\gamma_n(u), \gamma_n(v))$ such that

$$\psi(\gamma_n(u)\gamma_n(v)) \in \{-2n, -n, 0, n, 2n\} \times \{-2n, -n, 0, n, 2n\}.$$

Now it is easy to see that inverting the homomorphism γ_n at each node of $\mathcal{D}'$

$$(\gamma_n(u), \gamma_n(v)) \mapsto (u, v)$$

gives a 2-decomposition of w. □

4 A String in MIX That Has No 2-Decomposition

By Lemmas 3 and 4, in order to prove that there is no head grammar for MIX, it suffices to exhibit a string in MIX that has no 2-decomposition. The following is such a string:

$$z = a^5 b^{14} a^{19} c^{29} b^{15} a^5.$$

In this section, we prove that the string z has no 2-decomposition.[7]

It helps to visualize strings in MIX as closed curves in a plane. If w is a string in MIX, by plotting the coordinates of $\psi(v)$ for each prefix v of w, we can represent w by a closed curve C together with a map $t\colon [0, |w|] \to C$. The representation of the string z is given in Figure 2.

Let us call a string $w \in \{a, b, c\}^*$ such that $\psi(w) \in [-2, 2] \times [-2, 2]$ *long* if w contains all three letters, and *short* otherwise. (If $\psi(w) \notin [-2, 2] \times [-2, 2]$, then w is neither short nor long.) It is easy to see that a short string w always satisfies

$$|w|_a \leq 4, \quad |w|_b \leq 4, \quad |w|_c \leq 2.$$

The maximal length of a short string is 6. (For example, a^4c^2 and b^4c^2 are short strings of length 6.) We also call a pair of strings (v_1, v_2) *long* (or *short*) if v_1v_2 is long (or short, respectively).

According to the definition of an n-decomposition, a leaf node in a 2-decomposition must be labeled by a short pair of strings. We call a 2-decomposition *normal* if the label of every internal node is long. Clearly, any 2-decomposition can be turned into a normal 2-decomposition by deleting all nodes that are descendants of nodes with short labels.

[7]This fact was first verified by the computer program accompanying this paper. The program, written in C, implements a generic, memoized top-down recognizer for the language $\{ w \in \text{MIX} \mid w \text{ has a 2-decomposition} \}$, and does not rely on any special properties of the string z.

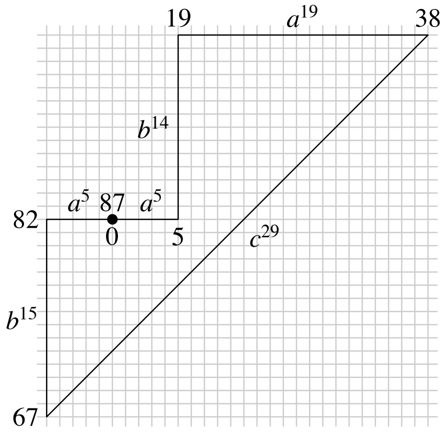


Figure 2: Graphical representation of the string $z = a^5b^{14}a^{19}c^{29}b^{15}a^5$. Note that every point (i, j) on the diagonal segment has $i > 7$ or $j < -2$.

One important property of the string z is the following:

Lemma 5. *If $z = x_1vx_2$ and $\psi(v) \in [-2, 2] \times [-2, 2]$, then either v or x_1x_2 is short.*

Proof. This is easy to see from the graphical representation in Figure 2. If a substring v of z has $\psi(v) \in [-2, 2] \times [-2, 2]$, then the subcurve corresponding to v must have initial and final coordinates whose difference lies in $[-2, 2] \times [-2, 2]$. If v contains all three letters, then it must contain as a substring at least one of $ba^{19}c$, $ac^{29}b$, and $cb^{15}a$. The only way to satisfy both these conditions is to have the subcurve corresponding to v start and end very close to the origin, so that x_1x_2 is short. (Note that the distance between the coordinate $(5, 0)$ corresponding to position 5 of z and the diagonal segment corresponding to the substring c^{29} is large enough that it is impossible for v to start at position 5 and end in the middle of c^{29} without violating the condition $\psi(v) \in [-2, 2] \times [-2, 2]$.) □

Lemma 5 leads to the following observation. Let us call a decomposition of a string *concatenation-free* if each of its non-leaf labels is the wrapping of the labels of the children.

Lemma 6. *If z has a 2-decomposition, then z has a normal, concatenation-free 2-decomposition.*

Proof. Let $\mathcal{D}$ be a 2-decomposition of z. Without loss of generality, we may assume that $\mathcal{D}$ is normal. Suppose that $\mathcal{D}$ contains a node μ whose label is the left or right concatenation of the labels of its children, (u_1, u_2) and (v_1, v_2). We only consider the case of left concatenation since the case of right concatenation is entirely analogous; so we suppose that the node μ is labeled by $(u_1u_2v_1, v_2)$. It follows that $z = x_1u_1u_2x_2$ for some x_1, x_2, and by Lemma 5, either u_1u_2 or x_1x_2 is short. If u_1u_2 is short, then the left child of μ is a leaf because $\mathcal{D}$ is normal. We can replace its label by (u_1u_2, ε); the label $(u_1u_2v_1, v_2)$ of μ will now be the wrapping (as well as left concatenation) of the two child labels, (u_1u_2, ε) and (v_1, v_2). If x_1x_2 is short, then we can combine by wrapping a single node labeled by (x_1, x_2) with the subtree of $\mathcal{D}$ rooted at the left child of μ, to obtain a new 2-decomposition of z. In either case, the result is a normal 2-decomposition of z with fewer instances of concatenation. Repeating this procedure, we eventually obtain a normal, concatenation-free 2-decomposition of z. □

Another useful property of the string z is the following:

Lemma 7. *Suppose that the following conditions hold:*

(i) $z = x_1u_1v_1yv_2u_2x_2$,

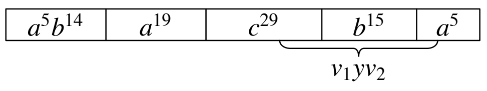
(ii) x_1yx_2 *is a short string, and*

(iii) *both* $\psi(u_1u_2)$ *and* $\psi(v_1v_2)$ *are in* $[-2, 2] \times [-2, 2]$.

Then either (u_1, u_2) *or* (v_1, v_2) *is short.*

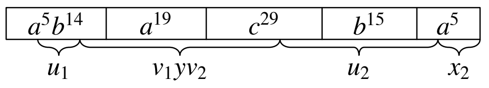
Proof. Suppose (u_1, u_2) and (v_1, v_2) are both long. Since (u_1, u_2) and (v_1, v_2) must both contain c, either u_1 ends in c and v_1 starts in c, or else v_2 ends in c and u_2 starts in c.

Case 1. u_1 ends in c and v_1 starts in c. Since (v_1, v_2) must contain at least one occurrence of a, the string v_1yv_2 must contain $cb^{15}a$ as a substring.



Since x_1yx_2 is short, we have $|y|_b \leq 4$. It follows that $|v_1v_2|_b \geq 11$. But v_1yv_2 is a substring of $c^{28}b^{15}a^5$, so $|v_1v_2|_a \leq 5$. This clearly contradicts $\psi(v_1v_2) \in [-2, 2] \times [-2, 2]$.

Case 2. v_2 ends in c and u_2 starts in c. In this case, $cb^{15}a^5$ is a suffix of u_2x_2. Since x_1yx_2 is short, $|x_2|_a \leq 4$. This means that $cb^{15}a$ is a substring of u_2 and hence $|u_2|_b = 15$.



On the other hand, since (v_1, v_2) must contain at least one occurrence of b, the string v_1yv_2 must contain $ba^{19}c$ as a substring. This implies that $|u_1u_2|_a \leq 10$. But since $|u_2|_b = 15$, we have $|u_1u_2|_b \geq 15$. This clearly contradicts $\psi(u_1u_2) \in [-2, 2] \times [-2, 2]$. □

We now assume that z has a normal, concatenation-free 2-decomposition $\mathcal{D}$ and derive a contradiction. We do this by following a certain path in $\mathcal{D}$. Starting from the root, we descend in $\mathcal{D}$, always choosing a non-leaf child, as long as there is one. We show that this path will never terminate.

The i-th node on the path will be denoted by μ_i, counting the root as the 0-th node. The label of μ_i will be denoted by $(w_{i,1}, w_{i,2})$. With each i, we associate three strings $x_{i,1}, y_i, x_{i,2}$ such that $x_{i,1}w_{i,1}y_iw_{i,2}x_{i,2} = z$, analogously to Lemma 1. Since $\psi(w_{i,1}w_{i,2}) \in [-2, 2] \times [-2, 2]$ and $\psi(z) = (0, 0)$, we will always have $\psi(x_{i,1}y_ix_{i,2}) \in [-2, 2] \times [-2, 2]$.

Initially, $(w_{0,1}, w_{0,2})$ is the label of the root μ_0 and $x_{0,1} = y_0 = x_{0,2} = \varepsilon$. If μ_i is not a leaf node, let $(u_{i,1}, u_{i,2})$ and $(v_{i,1}, v_{i,2})$ be the labels of the left and right children of μ_i, respectively. If the left child is not a leaf node, we let μ_{i+1} be the left child, in which case we have $(w_{i+1,1}, w_{i+1,2}) = (u_{i,1}, u_{i,2})$, $x_{i+1,1} = x_{i,1}$, $x_{i+1,2} = x_{i,2}$, and $y_{i+1} = v_{i,1}y v_{i,2}$. Otherwise, μ_{i+1} will be the right child of μ_i, and we have $(w_{i+1,1}, w_{i+1,2}) = (v_{i,1}, v_{i,2})$, $x_{i+1,1} = x_{i,1}u_{i,1}$, $x_{i+1,2} = u_{i,2}x_{i,2}$, and $y_{i+1} = y_i$.

The path $\mu_0, \mu_1, \mu_2, \ldots$ is naturally divided into two parts. The initial part of the path consists of nodes where $x_{i,1}y_ix_{i,2}$ is short. Note that $x_{0,1}y_0x_{0,2} = \varepsilon$ is short. As long as $x_{i,1}y_ix_{i,2}$ is short, $(w_{i,1}, w_{i,2})$ must be long and μ_i has two children labeled by $(u_{i,1}, u_{i,2})$ and $(v_{i,1}, v_{i,2})$. By Lemma 7, either $(u_{i,1}, u_{i,2})$ or $(v_{i,1}, v_{i,2})$ must be short. Since the length of z is 87 and the length of a short string is at most 6, exactly one of $(u_{i,1}, u_{i,2})$ and $(v_{i,1}, v_{i,2})$ must be long.

We must eventually enter the second part of the path, where $x_{i,1}y_i x_{i,2}$ is no longer short. Let μ_m be the first node belonging to this part of the path. Note that at μ_m, we have $\psi(x_{m,1}y_m x_{m,2}) = \psi(x_{m-1,1}y_{m-1}x_{m-1,2}) + \psi(v)$ for some short string v. (Namely, $v = u_{m-1,1}u_{m-1,2}$ or $v = v_{m-1,1}v_{m-1,2}$.)

Lemma 8. *If u and v are short strings and $\psi(uv) \in [-2,2] \times [-2,2]$, then $|uv|_d \leq 4$ for each $d \in \{a,b,c\}$.*

Proof. Since u and v are short, we have $|u|_a \leq 4, |u|_b \leq 4, |u|_c \leq 2$ and $|v|_a \leq 4, |v|_b \leq 4, |v|_c \leq 2$. It immediately follows that $|uv|_c \leq 4$. We distinguish two cases.

Case 1. $|uv|_c \leq 2$. Since $\psi(uv) \in [-2,2] \times [-2,2]$, we must have $|uv|_a \leq 4$ and $|uv|_b \leq 4$.

Case 2. $|uv|_c \geq 3$. Since $|u|_c \leq 2$ and $|v|_c \leq 2$, we must have $|u|_c \geq 1$ and $|v|_c \geq 1$. Also, $\psi(uv) \in [-2,2] \times [-2,2]$ implies that $|uv|_a \geq 1$ and $|uv|_b \geq 1$. Since u and v are short, it follows that one of the following two conditions must hold:

(i) $|u|_a \geq 1, |u|_b = 0$ and $|v|_a = 0, |v|_b \geq 1$.

(ii) $|u|_a = 0, |u|_b \geq 1$ and $|v|_a \geq 1, |v|_b = 0$.

In the former case, $|uv|_a = |u|_a \leq 4$ and $|uv|_b = |v|_b \leq 4$. In the latter case, $|uv|_a = |v|_a \leq 4$ and $|uv|_b = |u|_b \leq 4$. □

By Lemma 8, the number of occurrences of each letter in $x_{m,1}y_m x_{m,2}$ is in $[1,4]$. This can only be if

$$x_{m,1}x_{m,2} = a^j,$$
$$y_m = c^k b^l,$$

for some $j,k,l \in [1,4]$. This means that the string z must have been split into two strings $(w_{0,1}, w_{0,2})$ at the root of $\mathcal{D}$ somewhere in the vicinity of position 67 (see Figure 2).

It immediately follows that for all $i \geq m$, $w_{i,1}$ is a substring of $a^5b^{14}a^{19}c^{28}$ and $w_{i,2}$ is a substring of $b^{14}a^5$. We show by induction that for all $i \geq m$, the following condition holds:

(†) $ba^{19}c^{17}$ is a substring of $w_{i,1}$.

The condition (†) clearly holds for $i = m$. Now assume (†). Then $(w_{i,1}, w_{i,2})$ is long, and μ_i has left and right children, labeled by $(u_{i,1}, u_{i,2})$ and $(v_{i,1}, v_{i,2})$, respectively, such that $w_{i,1} = u_{i,1}v_{i,1}$ and $w_{i,2} = v_{i,2}u_{i,2}$. We consider two cases.

Case 1. $u_{i,1}$ contains c. Then $ba^{19}c$ is a substring of $u_{i,1}$. Since $u_{i,2}$ is a substring of $b^{14}a^5$, it cannot contain any occurrences of c. Since $\psi_1(u_{i,1}u_{i,2}) \in [-2,2]$, it follows that $u_{i,1}$ must contain at least 17 occurrences of c; hence $ba^{19}c^{17}$ is a substring of $u_{i,1}$. Since $(u_{i,1}, u_{i,2})$ is long, $(w_{i+1,1}, w_{i+1,2}) = (u_{i,1}, u_{i,2})$. Therefore, the condition (†) holds with $i+1$ in place of i.

Case 2. $u_{i,1}$ does not contain c. Then $(u_{i,1}, u_{i,2})$ is short and $(w_{i+1,1}, w_{i+1,2}) = (v_{i,1}, v_{i,2})$. Note that $v_{i,1}$ must contain at least 17 occurrences of c, but $v_{i,2}$ is a substring of $b^{14}a^5$ and hence cannot contain more than 14 occurrences of b. Since $\psi_2(v_{i,1}v_{i,2}) \in [-2,2]$, it follows that $v_{i,1}$ must contain at least one occurrence of b. Therefore, $ba^{19}c^{17}$ must be a substring of $v_{i,1} = w_{i+1,1}$, which shows that (†) holds with $i+1$ in place of i.

We have proved that (†) holds for all $i \geq m$. It follows that for all i, μ_i has two children and hence μ_{i+1} is defined. This means that the path $\mu_0, \mu_1, \mu_2, \ldots$ is infinite, contradicting the assumption that $\mathcal{D}$ is a 2-decomposition of z.

We have proved the following:

Lemma 9. *There is a string in MIX that has no 2-decomposition.*

Theorem 10. *There is no head grammar G such that $L(G) = MIX$.*

Proof. Immediate from Lemmas 3, 4, and 9. □

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