Multi-dimensional Temporal Logic for Events and States

Satoshi Tojo

JAIST
Asahidai 1-1, Nomi, Ishikawa 923-1292, Japan
e-mail: `tojo@jaist.ac.jp`

Abstract

The inclusion relation in temporal extents can be regarded as a prime requisite for the temporal expression. We propose a multi-dimensional temporal logic, combining the inclusion relation with the conventional precedence relation. First we define the syntax and the semantics of the fused logic, and then we apply the logic to the classification of occurrences to events and states, based on their upward/downward heredities. Thereafter, we consider the mutual relation between the precedence and the inclusion relations and discuss a proper set of axioms for the realistic time.

1 Introduction

In this paper, we introduce a multi-dimensional modal logic to represent the temporal structure of events and states. The linear temporal logic is the prime method of representation of time in natural language semantics. Another conventional approach, the interval-based time, has also contributed to the analysis of time in which two intervals are related in such ways that one overlaps the other, one includes the other, and so on [6,14,4]. Among such relations, van Benthem [15] regarded the inclusion relation as important, and defined the modalities $\Box^{\uparrow}$ and $\Box_{\downarrow}$, each of which represents 'all the superintervals' and the latter 'all the subintervals,' respectively.[1] In this paper, we further develop the idea and discuss the logic of the inclusion relation together with the conventional precedence.

Here, we do not consider the internal structure of each interval, and identify a time point with a shorter interval; for fear that the term 'interval' might be misunderstood as a sequence of time points, we employ the word *temporal extent*, that is a certain consecutive duration of time, in this study.

The temporal logic is formalized by such modal operators as F, G, P, and H, each of which represents 'some future,' 'all the future,' 'some past,' and 'all

[1] In the original literature [15], they are written as $\Box^{up}$ and $\Box_{down}$.

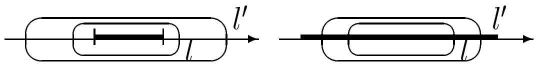

Fig. 1. upwar/downward hereditary

the past,' respectively. We add $\Box^{\uparrow}$ and $\Box_{\downarrow}$ to these, and propose a polymodal logic with regard to the ordinary temporal order and the inclusion relation. With this logic, we express the distinction of events and states.

In the following Section 2, we explain the intrinsic distinction of events ans states in terms of temporal heredity. In Section 3 we show the syntax and semantics of the logic. We apply the logic to events and states in Section 4, and thereafter, we discuss a proper set of axioms for the logic in Section 5. In Section 6 we summarize our contribution.

2 Upward/downward heredity

Let us consider an example of a simple detective story. If a murder suspect has an alibi between 2:00AM and 4:00AM, then (s)he has one also between 2:30AM and 3:30AM. Because 'have an alibi' is also valid in all the subintervals, this statement is said to be *downward hereditary.* On the contrary, if the presumed time of the death is between 3:00AM and 4:00AM, then it is also true that the victim died between 2:00AM and 5:00AM. As 'presumed death time' also holds in all the superintervals, the statement is said to be *upward hereditary* [11].

This distinction can be reduced to the following issue; if an *event* occurs in a point-wise instant, it becomes upward hereditary. Whereas, if a *state* persists with a certain duration of time, it naturally becomes downward hereditary.

The situation is explained in Fig. 1. An event is the perfective[2] view, in which whole the event structure including the beginning point and the culmination point is packed to a sole time point. As in the left-hand side of Fig. 1, if a temporal extent l includes this occurrence, so does l' ($\supseteq l$). On the contrary, if some state persists for a given temporal extent l' as in the right-hand side of the figure, then so does l ($\subseteq l'$). Hereafter, in case $l \subseteq l'$, l' is called to be a super-extent of l, and l is a sub-extent of l'. In this paper, we simply call those which are upward hereditary *events*, and those which are downward hereditary *states*, while we call both of them generically *occurrences.* If we claim that an occurrence φ is an event,

$$l \Vdash \varphi \text{ implies } l' \Vdash \varphi \ (l \subseteq l'), \tag{1}$$

and if an occurrence ψ is a state,

$$l' \Vdash \psi \text{ implies } l \Vdash \psi \ (l \subseteq l'). \tag{2}$$

[2] Note that the perfective view of an occurrence disregards its internal structure and renders the whole as one instant, which is different from the perfect aspect [1].

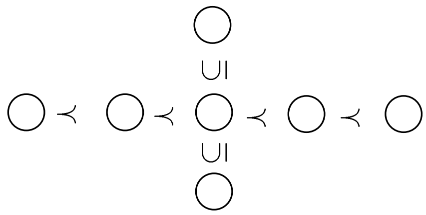

Fig. 2. Two-dimensional time

In the following section, we introduce the two-dimensional time. One dimension is the conventional precedence ($\prec$), and the other is the inclusion relation ($\subseteq$); both of which are given between two temporal extents so that they are arranged in the planar space as in Fig. 2.

3 Syntax and semantics of $K_{T\Box}$

In this section, we first give the syntax of the logic, and later, we give its Kripke semantics.

3.1 Syntax

The language consists of propositional variables $\varphi, \psi, \cdots$, logical connectives $\neg$, $\vee$, $\wedge$, $\Rightarrow$, and modal operators G, H, $\Box^{\uparrow}$, $\Box_{\downarrow}$ where parentheses and punctuation marks are added if necessary.

Modal operators are interpreted in the following way.

$G\varphi$ at all the future time, φ

$H\varphi$ at all the past time, φ

$\Box^{\uparrow}\varphi$ in all the super-extents, φ

$\Box_{\downarrow}\varphi$ in all the sub-extents, φ

Modal operators F, P, $\Diamond^{\uparrow}$, and $\Diamond_{\downarrow}$ are abbreviations of $\neg G \neg$, $\neg H \neg$, $\neg \Box^{\uparrow} \neg$, and $\neg \Box_{\downarrow} \neg$, respectively.

Note that the two temporal extents in the precedence relation do not share a common time ($t \not\prec t$). $F\Diamond^{\uparrow}\varphi$ and $\Diamond^{\uparrow}F\varphi$ are differently valuated; the former refers to some future of a super-extent of the current time while the latter does to a super-extent of some future. The former does not include the current time though the latter may include it. Thus, the truth values may be different in the two sentences. Generally speaking, all these operators are not commutative.

A modal logic with the modality $\Box$ is *normal* if (i) the logic includes all the tautologies, (ii) is closed under Modus Ponens, and (iii) satisfies the following property:

$$(K) \quad \Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi),$$

and (iv) if φ is a sentence of the logic so is $\Box\varphi$. Because all the $G, H, \Box^{\uparrow}, \Box_{\downarrow}$ satisfy the above conditions, the combined logic is *normal.*

First, we introduce the set of axioms for G and H and that of $\Box^{\uparrow}$ and $\Box_{\downarrow}$ independently. The logic K_T, the minimal tense logic, has the following axioms.

$$(4_{\rightarrow}) \quad G\varphi \Rightarrow GG\varphi \qquad (4_{\leftarrow}) \quad H\varphi \Rightarrow HH\varphi$$
$$(C_{\rightarrow}) \quad \varphi \Rightarrow GP\varphi \qquad (C_{\leftarrow}) \quad \varphi \Rightarrow HF\varphi$$

while the logic $K_{\Box}$ includes:

$$(4_{\uparrow}) \quad \Box^{\uparrow}\varphi \Rightarrow \Box^{\uparrow}\Box^{\uparrow}\varphi \qquad (4_{\downarrow}) \quad \Box_{\downarrow}\varphi \Rightarrow \Box_{\downarrow}\Box_{\downarrow}\varphi$$
$$(C_{\uparrow}) \quad \varphi \Rightarrow \Box^{\uparrow}\Diamond_{\downarrow}\varphi \qquad (C_{\downarrow}) \quad \varphi \Rightarrow \Box_{\downarrow}\Diamond^{\uparrow}\varphi$$
$$(T_{\uparrow}) \quad \Box^{\uparrow}\varphi \Rightarrow \varphi \qquad (T_{\downarrow}) \quad \Box_{\downarrow}\varphi \Rightarrow \varphi$$

The logic $K_T + K_{\Box}$ is the *fusion* of K_T and $K_{\Box}$,[3] and we denote it as $K_{T\Box}$ hereafter.

3.2 Kripke semantics

We introduce Kripke semantics for $K_{T\Box}$. A Kripke model for the logic is a tuple $\langle W, \prec, \succ, \subseteq, \supseteq, \Vdash \rangle$, where W is a non-empty set of possible worlds, and $\prec$ and $\subseteq$ are binary relations on W. Thus, each temporal extent is regarded as a possible world, and both of the precedence and the inclusion are two different accessibilities between the possible worlds. The semantics, i.e., $\Vdash$ is defined inductively as follows.

$$u \Vdash \varphi \wedge \psi \quad \textit{iff} \quad u \Vdash \varphi \text{ and } u \Vdash \psi,$$
$$u \Vdash \varphi \vee \psi \quad \textit{iff} \quad u \Vdash \varphi \text{ or } u \Vdash \psi,$$
$$u \Vdash \varphi \Rightarrow \psi \quad \textit{iff} \quad u \Vdash \varphi \text{ implies } u \Vdash \psi,$$
$$u \Vdash \neg\varphi \quad \textit{iff} \quad u \nVdash \varphi,$$
$$u \Vdash G\varphi \quad \textit{iff} \quad \forall v \in W, u \prec v \text{ implies } v \Vdash \varphi,$$
$$u \Vdash H\varphi \quad \textit{iff} \quad \forall v \in W, u \succ v \text{ implies } v \Vdash \varphi,$$
$$u \Vdash \Box^{\uparrow}\varphi \quad \textit{iff} \quad \forall v \in W, u \subseteq v \text{ implies } v \Vdash \varphi,$$
$$u \Vdash \Box_{\downarrow}\varphi \quad \textit{iff} \quad \forall v \in W, u \supseteq v \text{ implies } v \Vdash \varphi.$$

In Kripke semantics, $(4_{\rightleftarrows})$ and $(4_{\uparrow\downarrow})$ represent the transitivity, $(C_{\rightleftarrows})$ and $(C_{\uparrow\downarrow})$ the conversion, and $(T_{\uparrow\downarrow})$ the reflexivity, respectively. A formula φ is *true in model* $\mathcal{M}$, denoted by $\mathcal{M} \models \varphi$, if $u \Vdash \varphi$ for every $u \in W$. Now, we

[3] Let L_1 and L_2 be two modal logics. If L_1 is axiomatized by the set of axioms A_1 and L_2 is axiomatized by A_2, then the fusion $L_1 + L_2$ is axiomatized by the union $A_1 \cup A_2$ [5,2].

define the veridicality as follows.

$$\mathcal{M} \models G\varphi \Rightarrow GG\varphi \quad \textit{iff} \quad \forall u, v, w[u \prec v \wedge v \prec w \to u \prec w],$$
$$\mathcal{M} \models H\varphi \Rightarrow HH\varphi \quad \textit{iff} \quad \forall u, v, w[w \succ v \wedge v \succ u \to w \succ u],$$
$$\mathcal{M} \models \varphi \Rightarrow GP\varphi \quad \textit{iff} \quad \forall u, v[u \prec v \to v \succ u],$$
$$\mathcal{M} \models \varphi \Rightarrow HF\varphi \quad \textit{iff} \quad \forall u, v[u \succ v \to v \prec u],$$
$$\mathcal{M} \models \Box^{\uparrow}\varphi \Rightarrow \Box^{\uparrow}\Box^{\uparrow}\varphi \quad \textit{iff} \quad \forall u, v, w[u \subseteq v \wedge v \subseteq w \to u \subseteq w],$$
$$\mathcal{M} \models \Box_{\downarrow}\varphi \Rightarrow \Box_{\downarrow}\Box_{\downarrow}\varphi \quad \textit{iff} \quad \forall u, v, w[u \supseteq v \wedge v \supseteq w \to u \supseteq w],$$
$$\mathcal{M} \models \varphi \Rightarrow \Box^{\uparrow}\Diamond_{\downarrow}\varphi \quad \textit{iff} \quad \forall u, v[u \subseteq v \to v \supseteq u],$$
$$\mathcal{M} \models \varphi \Rightarrow \Box_{\downarrow}\Diamond^{\uparrow}\varphi \quad \textit{iff} \quad \forall u, v[u \supseteq v \to v \subseteq u],$$
$$\mathcal{M} \models \Box^{\uparrow}\varphi \Rightarrow \varphi \quad \textit{iff} \quad \forall u[u \subseteq u],$$
$$\mathcal{M} \models \Box_{\downarrow}\varphi \Rightarrow \varphi \quad \textit{iff} \quad \forall u[u \supseteq u].$$

If $\prec$ and $\subseteq$ satisfy all of the above conditions for $\mathcal{M}$, $\mathcal{M}$ is called to be a $K_{T\Box}$-model. Now, we can construct the canonical model [3]; i.e., $\forall\varphi$, $\varphi \notin K_{T\Box}$ *iff* there exists $K_{T\Box}$-model $\mathcal{M}$ such that $\mathcal{M} \not\models \varphi$ (completeness).

4 Events and states

4.1 Downward heredity

A proposition is *gestalt* if it never holds over two temporal extents one of which properly contains the other [11]. That is, the gestalt of an occurrence shows the exact temporal extent where the occurrence takes place on the time axis.

If we directly translate the feature of a state, (2), into a formula,

$$(2)' \quad \varphi \Rightarrow \Box_{\downarrow}\varphi.$$

Now let us consider the possibility that there exists a super-extent ($\Diamond^{\uparrow}$), in all the sub-extents of which φ still holds ($\Box_{\downarrow}\varphi$). In Fig. 3, if we reside in t_1 and φ is a state ($\varphi \Rightarrow \Box_{\downarrow}\varphi$), we can assume an enlarged temporal extent t_2 ($\supseteq t_1$) where $t_2 \Vdash \Box_{\downarrow}\varphi$. In case φ does not hold before and after the enlarged extent t_2, i.e.,

$$t_2 \Vdash H\neg\varphi \wedge \Box_{\downarrow}\varphi \wedge G\neg\varphi,$$

then the enlarged extent specified by t_2 can be regarded as the *maximal* extent, i.e., the gestalt of the state.

Example 1 *If "Alice was sleeping between 1*PM *and 2*PM*" then we can infer that she has slept in any sub-extent of [1*PM*, 2*PM*], as:*

$$[1{:}30_{PM}, 2_{PM}] \Vdash \langle\!\langle A \textit{ sleeps} \rangle\!\rangle,$$

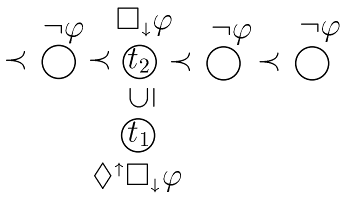

Fig. 3. Maximal duration

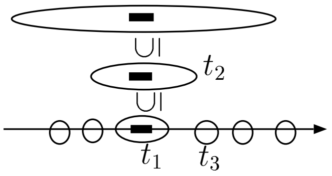

Fig. 4. Upward heredity

where A stands for 'Alice.' In this case, we can also infer that there must be a maximal extent, including [1PM, 2PM], in any sub-extents of which she sleeps.

$$[\mathit{1}_{PM}, \mathit{2}_{PM}] \Vdash \Diamond^{\uparrow}\Box_{\downarrow}\langle\!\langle A\ sleeps\rangle\!\rangle.$$

If she actually took a siesta between 12:30PM and 3PM,

$$[\mathit{12{:}30}_{PM}, \mathit{3}_{PM}] \Vdash H\neg\langle\!\langle A\ sleeps\rangle\!\rangle \wedge \Box_{\downarrow}\langle\!\langle A\ sleeps\rangle\!\rangle \wedge G\neg\langle\!\langle A\ sleeps\rangle\!\rangle.$$

Here, $\langle\!\langle\ \rangle\!\rangle$ is an identical event which happens once and for all, and is not a situation type.

4.2 Upward heredity

On the contrary, if an occurrence of an event is upward hereditary,

$$(1)' \quad \varphi \Rightarrow \Box^{\uparrow}\varphi,$$

and in this case, φ should not appear both in the past and in the future. This situation is depicted in Fig. 4. The thick line is the gestalt of the event and is encircled by its temporal extent. Let t_1 be the original temporal extent of φ and t_2 $(\supseteq t_1) \Vdash \varphi$; if $t_1 \prec t_3$, necessarily $t_3 \nVdash \varphi$ even though t_3 may be included in t_2.

However, in the similar way to the downward heredity, we can assume the minimal extent in which the event occurred, as:

$$\Diamond_{\downarrow}(H\neg\varphi \wedge \Box^{\uparrow}\varphi \wedge G\neg\varphi).$$

Example 2 *If "Betty woke up between 7:30AM and 8AM" then we can infer that there must be the minimal extent in [7:30AM, 8AM], that could be more adequately called an instant, when she got up. Let u be such a small extent, then:*

$$u \Vdash H\neg\langle\!\langle B\ wakes\ up\rangle\!\rangle \wedge \Box^{\uparrow}\langle\!\langle B\ wakes\ up\rangle\!\rangle \wedge G\neg\langle\!\langle B\ wakes\ up\rangle\!\rangle.$$

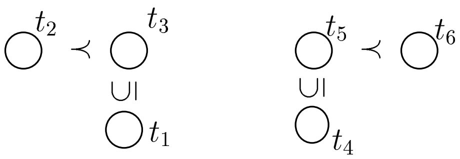

Fig. 5. Triangular constraints

5 Triangular constraints

The two accessibilities in Section 3 cannot be independent of each other for a realistic model of time. Here, a model of time means an empirically plausible time, or in other words, all the given temporal extents can be mapped consistently onto the physical time axis, that is linear and unbounded.

As is the left-hand side of Fig. 5, given three temporal extents t_1, t_2 and t_3, if t_1 is included in t_3 and t_2 precedes t_3, then t_2 should also precede t_1. The figure suggests the following three constraints. First, seen from t_2,

$$F\Diamond_{\downarrow}\varphi \Rightarrow F\varphi, \tag{3}$$

that is, if $t_2 \Vdash F\Diamond_{\downarrow}\varphi$ then $t_3 \Vdash \Diamond_{\downarrow}\varphi$, and thus $t_1 \Vdash \varphi$. Next, if we see from t_3, then:

$$P\varphi \Rightarrow \Box_{\downarrow}P\varphi. \tag{4}$$

Namely, if $t_3 \Vdash P\varphi$ then $t_3 \Vdash \Box_{\downarrow}P\varphi$, i.e., for all $t_1 \subseteq t_3$, $t_1 \Vdash P\varphi$. Finally, if we see from t_1, then:

$$\Diamond^{\uparrow}P\varphi \Rightarrow P\varphi. \tag{5}$$

This means that if $t_1 \Vdash \Diamond^{\uparrow}P\varphi$ then $t_3 \Vdash P\varphi$ and thus $t_2 \Vdash \varphi$.

Because the axioms (3), (4), and (5) concerns the same arrangement of three temporal extents, the meanings should be equivalent. Actually, we can show that the three axioms are the identical one (see Appendix). We name the axiom (Δ_1). The *duals* [4] of them become:

$$(3)^* \; G\varphi \Rightarrow G\Box_{\downarrow}\varphi,$$
$$(4)^* \; \Diamond_{\downarrow}H\varphi \Rightarrow H\varphi,$$
$$(5)^* \; H\varphi \Rightarrow \Box^{\uparrow}H\varphi.$$

In the very similar way, for the right-hand side of Fig. 5 we can show that the following conditions represent the same axiom. If $t_4 \Vdash \Diamond^{\uparrow}F\varphi$, then $t_5 \Vdash F\varphi$ and $t_6 \Vdash \varphi$.

$$\Diamond^{\uparrow}F\varphi \Rightarrow F\varphi. \tag{6}$$

If $t_5 \Vdash F\varphi$, then $t_5 \Vdash \Box_{\downarrow}F\varphi$.

$$F\varphi \Rightarrow \Box_{\downarrow}F\varphi. \tag{7}$$

[4] The dual is the contraposition of the original formula, the propositional symbols of which are replaced for its negatives, and is denoted by $(*)$.

If $t_6 \Vdash P\Diamond_{\downarrow}\varphi$, then $t_5 \Vdash \Diamond_{\downarrow}\varphi$ and $t_4 \Vdash \varphi$.

$$P\Diamond_{\downarrow}\varphi \Rightarrow P\varphi. \tag{8}$$

The followings are the another family of axioms. We name it (Δ_2). The dual of them become:

$$(6)^* \ G\varphi \Rightarrow \Box^{\uparrow}G\varphi,$$
$$(7)^* \ \Diamond_{\downarrow}G\varphi \Rightarrow G\varphi,$$
$$(8)^* \ H\varphi \Rightarrow H\Box_{\downarrow}\varphi.$$

Combining $(T_{\uparrow\downarrow})$ with the above formulae, we obtain the following equations of modalities.

$$G \equiv \Box^{\uparrow}G \equiv \Diamond_{\downarrow}G$$
$$H \equiv \Box^{\uparrow}H \equiv \Diamond_{\downarrow}H$$
$$F \equiv \Box_{\downarrow}F \equiv \Diamond^{\uparrow}F$$
$$P \equiv \Box_{\downarrow}P \equiv \Diamond^{\uparrow}P$$

At this stage, the following set of axioms:

$$K_{T\Box\Delta} = K_{T\Box} + \{\Delta_1, \Delta_2\}$$

can be considered as a proper candidate of the two-dimensional temporal structure.

6 Discussion

We have proposed a multi-dimensional temporal logic, combining the logic $K_{\Box}$ of the inclusion relation $(\subseteq)$ of temporal extents together with the logic K_T of the conventional precedence relation $(\prec)$, and showed the syntax and the semantics of the fusion of them as $K_{T\Box}$. With this logic, we gave explanations for the gestalt of occurrences in terms of temporal extents, as well as the progressive and the perfect aspects. Thereafter, we also added several axioms to constrain the relationship between two different accessibilities, $\subseteq$ and $\prec$, to express the realistic time.

Though we have mainly discussed the distinction of upward/ downward heredity in this paper, we can extend the notion to the classification of aspects. In [15], the progressive and the perfective aspects were represented by $\Diamond^{\uparrow}$ and P, respectively. Although these may look rather oversimplified, we can support the idea in that both of the progressive form $\Diamond^{\uparrow}\varphi$ and the perfect form $P\varphi$ are downward hereditary $(\Box_{\downarrow})$, as: $\Diamond^{\uparrow}\varphi \ \vdash_{(C_{\downarrow})} \ \Box_{\downarrow}\Diamond^{\uparrow}\Diamond^{\uparrow}\varphi \ \vdash_{(4_{\uparrow})^*} \ \Box_{\downarrow}\Diamond^{\uparrow}\varphi$, and $P\varphi \Rightarrow \Box_{\downarrow}P\varphi$ (See (4) in Section 5).

In the similar way to the conventional temporal logic, a set of axioms specifies a kind of multi-dimensional temporal logic. Namely, adding or subtracting some axioms, we can represent different temporal structures. At the current stage, $K_T + \{(4_{\uparrow\downarrow}), (T_{\uparrow\downarrow})\}$, i.e., $K_{T\Box} - \{(C_{\uparrow\downarrow})\}$ was proved to be *decidable* as in [17]. The sequent rules for K_T were given in [9] and those for $\{(4), (T)\}$ with (K) becomes S4. Although K_T+S4 cannot satisfy the cut elimination property [7], we can employ the restricted cut elimination for the subformulae [12,13]. Because the restricted sequent system satisfies the subformula property, we can show that the whole sequent system also satisfies it. If a system has the subformula property, it has a finite model. According to Harrop's theorem [3], if a system has a finite model with a finite set of axioms, it is decidable.

On the contrary, because the system $\{(4_{\uparrow\downarrow}), (C_{\uparrow\downarrow})\}$ is same as K_T, $K_{T\Box} - \{(T_{\uparrow\downarrow})\}$ is the fusion of two K_T's and again becomes decidable.

Although we can claim that $K_{T\Box\Delta}$ gives the proper relationship of the inclusion and the precedence, the sequent system and the proof method of which would become more complicated. The logical features such as decidability of the extended multi-dimensional modal logic, as well as the proof system, are under investigation.

References

[1] B. Comrie. *Aspect*, Cambridge University Press, 1976.

[2] D. M. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics: theory and applications*, Studies in logic and the foundations of mathematics, vol. 148, Elsevier, 2003.

[3] R. Goldblatt. *Logics of Time and Computation*, Second Edition, CSLI Lecture Note No.7, Center for the Study of Language and Information, Stanford University, 1992.

[4] H. Kamp and U. Reyle. *From Discourse to Logic*, Kluwer Academic Publisher's, 1993.

[5] M. Krachat and F. Wolter. Properties of independently axiomatizable bimodal logics, *Journal of Symbolic Logic 56*, pp. 1469-1485, 1991.

[6] F. Landman. *Structures for Semantics*, Kluwer Academic Publisher's, 1991.

[7] A. Maruyama, S. Tojo and H. Ono. Decidability of temporal epistemic logics for multi-agent models, *Proceedings of the ICLP'01 Workshop on Computational Logic in Multi-Agent Systems (CLIMA-01)*, pp.31-40, 2001.

[8] M. Moens and M. Steedman. Temporal ontology and temporal reference, *Computational Linguistics*, 14(2), pp.15–28, 1988.

[9] H. Nishimura. *A study of some tense logics by Gentzen's sequential method*, Publications of the Research Institute for Mathematical Sciences, Kyoto University 16, pp.343-353, 1990.

[10] T. Parsons. : *Events in the Semantics of English*, MIT press, 1990.

[11] Y. Shoham, *Reasoning about Change*, The MIT Press, 1988.

[12] M. Takano. Subformula property as a substitute for cut-elimination in modal propositional logics, *Mathematica japonica Vol.37*, pp.1129-1145, 1992.

[13] M. Takano. A modified subformula property for the modal logics K5 and K5D, bulletin of Section of Logic, Vol. 30, pp.67-70, 2001.

[14] J. van Benthem. *The Logic of Time - second edition*, Kluwer Academic Press, 1991.

[15] J. van Benthem. *Epistemic and temporal reasoning*, edited by Dov M. Gabbay, C.J. Hogger and J.A. Robinson, Oxford, Clarendon Press, Handbook of logic in artificial intelligence and logic programming, vol. 4, pp. 292-296, 1995.

[16] Z. Vendler. Linguistics in Philosophy, Cornell University Press, 1967.

[17] S. Yoshioka and S. Tojo. Many-dimensional Modal Logic of Tense and Temporal Interval and its Decidability, WEC 2005.

Appendix

We here prove the equality of (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

- (3) $\vdash$ (4): By $(C_{\leftarrow})$ $\Diamond_{\downarrow}\varphi \Rightarrow HF(\Diamond_{\downarrow}\varphi)$, and by (3) $\Diamond_{\downarrow}\varphi \Rightarrow HF\varphi$. Its dual becomes $PG\varphi \Rightarrow \Box_{\downarrow}\varphi$, and thus, $PG(P\varphi) \Rightarrow \Box_{\downarrow}P\varphi$. From $(C_{\rightarrow})$, $P\varphi \Rightarrow PGP\varphi$; hence, $P\varphi \Rightarrow \Box_{\downarrow}P\varphi$ (4).
- (4) $\vdash$ (3): By $(C_{\rightarrow})$, $G\varphi \Rightarrow GP(G\varphi)$. From (4) $P(G\varphi) \Rightarrow \Box_{\downarrow}P(G\varphi)$, and by $(C^{*}_{\leftarrow})$, $PG\varphi \Rightarrow \Box_{\downarrow}PG\varphi \Rightarrow \Box_{\downarrow}\varphi$. Therefore, $GPG\varphi \Rightarrow G\Box_{\downarrow}\varphi$. Combining them, we obtain $G\varphi \Rightarrow G\Box_{\downarrow}\varphi$. Its dual becomes $F\Diamond_{\downarrow}\varphi \Rightarrow F\varphi$.
- (5) $\vdash$ (4): By $(C_{\downarrow})$, $P\varphi \Rightarrow \Box_{\downarrow}\Diamond^{\uparrow}P\varphi$. Given $\Diamond^{\uparrow}P\varphi \Rightarrow P\varphi$ (5), the right-hand side of the above is reduced to $\Box_{\downarrow}P\varphi$.
- (4) $\vdash$ (5): By $(C_{\uparrow})$, $H\varphi \Rightarrow \Box^{\uparrow}\Diamond_{\downarrow}H\varphi$. Because the dual of (4) becomes $\Diamond_{\downarrow}H\varphi \Rightarrow H\varphi$, we obtain $H\varphi \Rightarrow \Box^{\uparrow}H\varphi$. Its dual becomes $\Diamond^{\uparrow}P\varphi \Rightarrow P\varphi$.

We can show the equality of (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) similarly, replacing the modal operators symmetrically.