Metrical Grids and Generalized Tier Projection

Yiding Hao
Yale University
New Haven, CT, USA
`yiding.hao@yale.edu`

Abstract

This paper formalizes metrical grid theory (MGT, Prince, 1983; Hayes, 1995) and studies its expressive power. I show that MGT analyses of a certain form can describe stress systems beyond the *input tier-based input strictly local* functions proposed by Hao and Andersson (2019), but conjecture that such analyses do not describe systems beyond the *input tier-based strictly local* languages of Baek (2018). These results reveal fundamental differences between the three formalisms.

1 Introduction

The problem of unbounded stress has proven challenging for subregular phonology. Baek (2018) has recently shown that unbounded stress provides a counterexample to the *weak subregular hypothesis* (Heinz, 2018), which claims that phonological phenomena can be represented by *tier-based strictly local languages* (TSL, Heinz et al., 2011) when viewed as *decision problems* that accept grammatical utterances. To remedy this, Baek proposes that the decision to project a symbol to the tier may be conditioned by local contextual information such as the presence of word boundaries. This proposal has been extended to a *generalized tier projection* system in which tier projection is implemented by arbitrary deleting functions (Mayer and Major, 2018; Graf and Mayer, 2018).

Viewing unbounded stress as a *transduction problem* that maps underlying forms without prosodic representation to surface forms marked for primary stress, Hao and Andersson (2019) show that unbounded stress systems are similarly handeled by generalized tier projection, but that they fall outside the class of *tier-based input strictly local functions* (TISL, Chandlee, 2014). Hao and Andersson demonstrate that *default-to-opposite-side* (DO) systems can be captured by adapting

generalized tier projection to subregular functions. However, it turns out that the "bidirectional" nature of *default-to-same-side* (DS) systems cannot be implemented by subsequential functions, which allow only a single unidirectional pass over the input. They instead propose that DS systems should be understood as *interaction-free weakly deterministic functions* in the sense of McCollum et al. (2018).

These results raise conceptual questions regarding the treatment of stress in subregular phonology. In particular, the mechanism of generalized tier projection intuitively seems powerful and *ad-hoc*. The basic elements of subregular phonology, namely strict locality and the traditional tier projection system of Heinz et al. (2011), can be viewed as formalizations of rule and tier systems that are well-established in phonological theory (Chandlee, 2014; Chomsky and Halle, 1968; Goldsmith, 1976). While generalized tier projection enables Baek (2018) and Hao and Andersson (2019) to define classes of finite-state machines that capture unbounded stress, it is unclear whether generalized tier projection is similarly grounded in existing phonological constructs. Reflecting on this issue, Hao and Andersson observe that the generalized tier-projection mechanism they use to produce *Dybo's Rule* (Dybo, 1977), a model of the stress system in Abkhaz, bears a striking resemblance to the syllable tier used in a standard analysis of that system within *metrical grid theory* (MGT, Prince, 1983; Hayes, 1995). From that observation we might hypothesize that stress systems computed using generalized tier projection naturally correspond to those described by MGT.

In this paper, I will argue that this intuition does not hold for the transduction problem, though it may hold for the decision problem. To that end, I define a formal model of MGT in Section 4,

and show in Section 5 that the stress systems described by MGT do not correspond to those represented by functions defined using generalized tier projection. I also give evidence to suggest that decision problems described by MGT can be represented using generalized tier projection even when their corresponding transduction problems cannot. These results imply not only that functions based on generalized tier projection are not grounded in MGT, but also that the typological predictions they make about the range of possible stress systems differ from those made by MGT and by decision problems.

Technical definitions used in this paper are given in Section 2, and Section 3 reviews the existing results on unbounded stress in subregular phonology. Section 6 concludes.

2 Preliminaries

In this paper, uppercase Greek letters denote finite alphabets not including the boundary symbols $\rtimes$ and $\ltimes$. The length of a string x is denoted by $|x|$, and λ denotes the empty string. Alphabet symbols are identified with strings of length 1, and individual strings are identified with singleton sets of strings. For $k \geq 0$, α^k denotes α concatenated with itself k-many times, $\alpha^{<k}$ denotes $\bigcup_{i=0}^{k-1} \alpha^i$, α^* denotes $\bigcup_{i=0}^{\infty} \alpha^i$, and α^+ denotes $\alpha\alpha^*$. The *longest common prefix* of a set of strings A is the longest string $\mathrm{lcp}(A)$ such that every string in A begins with $\mathrm{lcp}(A)$.

For sets A and B, the notation $f : A \to B$ means that f is a function with domain A and codomain B. The *range of f* is the set $\{y | \exists x. f(x) = y\} \subseteq B$. A function $f : A \to B$ is *injective* if for every $x, y \in A$, $f(x) = f(y)$ if and only if $x = y$. A function $f : \Sigma^* \to \Gamma^*$ is *same-length* if and only if for all $x \in \Sigma^*$, $|f(x)| = |x|$.

A *subsequential finite-state transducer* (SFST) is a 6-tuple $T = \langle Q, \Sigma, \Gamma, q_0, \to, \omega \rangle$, where

- Q is the set of *states*, with $q_0 \in Q$ being the *start state*;

- Σ and Γ are the *input* and *output alphabets*, respectively;

- $\to : Q \times \Sigma \to Q \times \Gamma^*$ is the *transition function*; and

- $\omega : Q \to \Gamma^*$ is the *final output function*.

For $x \in \Sigma^*$; $y \in \Gamma^*$; and $q, r \in Q$, the notation $q \xrightarrow{x:y} r$ means that T emits y to the output stream

and transitions to state r if it reads x in the input stream while it is in state q. Letting $f : \Sigma^* \to \Gamma^*$, we say that T *computes* f if for every $x \in \Sigma^*$, $f(x) = y\omega(q)$, where $q_0 \xrightarrow{x:y} q$. A function is *subsequential* if it is computed by an SFST.

For a string $x \neq \lambda$, I use the following indexing notation.

- For $1 \leq i \leq j \leq |x|$, $x[i : j]$ is the substring of x such that $x = wx[i : j]y$, where $|w| = i - 1$ and $|y| = |x| - j$.

- For $-|x| \leq u, v \leq |x|$ and $1 \leq i \leq j \leq |x|$, $x[u : v] = x[i : j]$ if $u \equiv i \mod (|x| + 1)$ and $v \equiv j \mod (|x| + 1)$.

- For each i, $x[i] := x[i : i]$; $x[i :] := x[i : |x|]$; and $x[: i] := x[1 : i]$.

The remainder of this section reviews the algebraic characterization of subsequential functions as well as tier projection and strict locality.

2.1 Subsequential Functions

Independently of SFSTs, the subsequential functions can be characterized using two operations on string functions.

Definition 1. Let $f : \Sigma^* \to \Gamma^*$. We define the function $f^{\leftarrow} : \Sigma^* \to \Gamma^*$ by

$$f^{\leftarrow}(x) := \mathrm{lcp}\left(\{f(xy) | y \in \Sigma^*\}\right).$$

For any $x, y \in \Sigma^*$, $f_x^{\to}(y)$ denotes the string such that $f(xy) = f^{\leftarrow}(x)f_x^{\to}(y)$. We refer to $f_x^{\to}$ as the *translation of f by x* and to $f^{\leftarrow}$ as *f top*.

The translations of a subsequential function may be used to construct the minimal SFST for that function, analogously to the Nerode–Myhill construction for the minimal finite-state automaton of a regular language.

Theorem 2 (Raney, 1958). *A function $f : \Sigma^* \to \Gamma^*$ is subsequential if and only if the set $\{f_x^{\to} | x \in \Sigma^*\}$ is finite.*

For a subsequential function f with minimal SFST T, the translations of f are in bijection with the states of T. After reading input x, T outputs $f^{\leftarrow}(x)$ and enters the state corresponding to $f_x^{\to}$.

2.2 Homomorphisms

This paper will frequently make use of a class of functions known as *homomorphisms*.

Definition 3. A function $h : \Sigma^* \to \Gamma^*$ is a *homomorphism* if for every $x, y \in \Sigma^*$, $h(xy) = h(x)h(y)$.

Intuitively, homomorphisms are functions that replace each symbol of Σ with a string in Γ^*. As such, homomorphisms are completely determined by their values on the input alphabet.

Proposition 4. *Let* $h, g : \Sigma^* \to \Gamma^*$ *be homomorphisms. If* $h(x) = g(x)$ *for each* $x \in \Sigma$, *then* $h = g$.

2.3 Locality and Tier Projection

Tier projections are functions that delete certain symbols in an input string x. A tier τ can be used to enhance notions of locality defined by grammars, automata, and transducers by having local dependencies be enforced between adjacent symbols in $\tau(x)$ instead of x, effectively ignoring symbols deleted by τ.

Definition 5. A *tier projection* is a function $\tau : \Sigma^* \to \Sigma^*$ such that $\tau(\lambda) = \lambda$ and for all $x \in \Sigma^+$, $\tau(x) = y_1 y_2 \ldots y_{|x|}$, where for each i, y_i is either $x[i]$ or λ. If τ is a homomorphism, then we identify τ with the subset $\Delta \subseteq \Sigma$ such that for all $\delta \in \Delta$, $\tau(\delta) = \delta$.

Symbols not deleted by a tier projection are said to be *projected to the tier*. Tier-based strictly local functions are defined to be functions computed by minimal SFSTs whose states record the most recent $k - 1$ symbols projected to some tier, for some $k > 0$. In this paper, we assume that the states only record symbols from the SFST input projected to the tier; variants of the definitions below where the tier projects symbols of the output have also been defined (Chandlee, 2014; Chandlee et al., 2015; Burness and McMullin, 2019).

Definition 6. Let $k > 0$, and let $\tau : \Sigma^* \to \Sigma^*$ be a tier projection. A function $f : \Sigma^* \to \Gamma^*$ is *generalized input strictly k-local on tier τ* (k-GTISL on tier τ) if for all $x, y \in \Sigma^*$,

$$\tau^{\leftarrow}(x)[: 1 - k] = \tau^{\leftarrow}(y)[: 1 - k]$$

implies $f_x^{\rightarrow} = f_y^{\rightarrow}$. We say that f is

- *input strictly k-local* (k-ISL) if τ is the identity function;[1]

---

[1]In the automata theory literature, k-ISL functions are known as k-local functions (Vaysse, 1986). See Sakarovitch (2009, pp. 661–664) for an overview.

- *input strictly k-local on tier τ* (k-TISL on tier τ) if τ is a homomorphism; and

- *j-input strictly k-local on tier τ* (j-I-k-TISL on tier τ) if τ is j-TISL.

Remark 7. Homomorphisms are 1-ISL functions.

Tier-based strictly local languages are defined to be sets of strings whose images under some tier projection only contain substrings deemed permissible.

Definition 8. Let $k > 0$, and let $\tau : \Sigma^* \to \Sigma^*$ be a tier projection. A language $L \subseteq \Sigma^*$ is *generalized strictly k-local on tier τ* (k-GTSL on tier τ) if there exists $S \subseteq (\Sigma \cup \{\rtimes, \ltimes\})^k$ such that for all $x \in \Sigma^*$, $x \in L$ if and only if every length-k substring of $\rtimes^{k-1} \tau(x) \ltimes^{k-1}$ is in S. We say that L is

- *strictly k-local* (k-SL) if τ is the identity function;

- *strictly k-local on tier τ* (k-TSL on tier τ) if τ is a homomorphism; and

- *j-input strictly k-local on tier τ* (j-I-k-TSL on tier τ) if τ is j-ISL.

3 Stress in Subregular Phonology

Stress is a phonological process in which syllables are assigned varying levels of prominence (i.e., primary stress, secondary stress, or no stress) with respect to one another. Stress is *culminative*, meaning that each word contains exactly one maximally-prominent syllable. Stress is usually represented by marking syllables with their prominence levels, leaving all other information about those syllables intact. This section introduces the formalism I use to represent stress and reviews the current results on stress in subregular phonology.

Throughout this paper, I treat syllables as atomic units, and I represent them using symbols drawn from an alphabet Σ. Words, being strings of syllables, are elements of Σ^*. When a syllable $\sigma \in \Sigma$ is assigned primary stress, I mark this syllable with a diacritic $\acute{\sigma}$. I do not mark syllables for secondary stress. Thus, the set $\acute{\Sigma} := (\Sigma \cup \{\acute{\sigma} | \sigma \in \Sigma\})^*$ is the complete alphabet of symbols used to discuss stress.

Using this representation, stress in a particular language can be construed in two ways. Firstly, we may think of a stress system as a function mapping words without stress marking to words with stress

marking. This formalizes the transduction problem for stress.

Definition 9. A *stress system* is a same-length function $s : \Sigma^* \to \acute{\Sigma}^*$ such that for every $x \in \Sigma^+$, there exists $i > 0$ and $\sigma \in \Sigma$ such that

- $x[i] = \sigma$ and $s(x)[i] = \acute{\sigma}$; and

- for all $j > 0$, if $j \neq i$, then $s(x)[j] = x[j]$.

Remark 10. All stress systems are injective.

Secondly, we can think of a stress system as the set of all words in which stress has been assigned correctly. This formalizes the decision problem for stress.

Definition 11. A *stress constraint* is any subset $C \subseteq \acute{\Sigma}$ such that C is the range of some stress system $s : \Sigma^* \to \acute{\Sigma}^*$.

These formalizations are "equivalent" in the sense that we can easily convert between them.

Definition 12. Let $s : \Sigma^* \to \acute{\Sigma}^*$ be a stress system. The *stress constraint given by* s is the range of s.

Definition 13. Let $C \subseteq \acute{\Sigma}^*$ be a stress constraint. The *stress system given by* C is the stress system $s_C : \Sigma^* \to \acute{\Sigma}^*$ whose range is C.

A well-known example of a stress system is the *leftmost heavy otherwise rightmost* (LHOR) system (Hayes, 1995). In this system, syllables are either *light* or *heavy*. The leftmost heavy syllable in a word receives primary stress. If there are no heavy syllables in a word, then the rightmost (light) syllable receives primary stress. In Kwak'wala, for example, heavy syllables are those that contain a long vowel or a vowel with a coda consisting of [m], [n], or [l] (Bach, 1975). Some illustrative examples are given below.

(14) LHOR stress in Kwak'wala (Boas et al., 1947; Bach, 1975)

    a. [tsˀəmaˈaːtud]   'to   melt   away something in the ear'

    b. [ˈbaːbagwəm] 'boys'

    c. [gəgəˈnəm] 'wives'

LHOR stress can be formalized using the following stress system. Heavy syllables are represented by the symbol H, while light syllables are represented by the symbol L.

Definition 15. Let $\Sigma := \{H, L\}$. The *LHOR system* is defined as follows. For $u \in L^*$, $v \in \Sigma$, and $w \in \Sigma^*$,

$$\text{LHOR}(uvw) = \begin{cases} u\acute{v}w, & v = H \\ uw\acute{v}, & uvw \in L^+. \end{cases}$$

It is easy to see that the transduction problem for LHOR is I-TSL but not TSL.

Proposition 16. LHOR *is not TSL.*

Proof. Fixing $k > 0$ and homomorphic tier τ, let us show that LHOR is not k-TSL on τ. Suppose $L \notin \tau$. Then, observe that $\text{suff}^{k-1}(\tau^{\leftarrow}(\lambda)) = \text{suff}^{k-1}(\tau^{\leftarrow}(L)) = \rtimes$, but $\text{LHOR}_\lambda^{\to}(\lambda) = \lambda$, while $\text{LHOR}_L^{\to}(\lambda) = \acute{L}$. Thus, LHOR is not k-TSL on τ if $L \notin \tau$. But if $L \in \tau$, then we have $\text{LHOR}_{HL^k}^{\to}(H) = H$ and $\text{LHOR}_{L^k}^{\to}(H) = L\acute{H}$ even though $\text{suff}^{k-1}(HL^k) = \text{suff}^{k-1}(L^k) = L^{k-1}$. Therefore, LHOR is also not k-TSL on τ if $L \in \tau$, so we conclude that it is not TSL. $\square$

Proposition 17. LHOR *is 2-I-2-TSL.*

Proof. Consult Figure 1. $\square$

Similarly, as Baek (2018) shows, the decision problem for LHOR is I-TSL but not TSL.

Proposition 18. C_{LHOR} *is not TSL.*

Proof. Fix $k > 0$, and suppose C_{LHOR} is k-TSL on homomorphic tier τ. It is clear that $\acute{H}, \acute{L} \in \tau$; otherwise, we would have $\acute{H}H, \acute{L}L \in C_{\text{LHOR}}$. Furthermore, we must have $H \in \Gamma$ and $L \in \Gamma$, since otherwise we would have $H\acute{H} \in C_{\text{LHOR}}$ and $\acute{L}L \in C_{\text{LHOR}}$, respectively. Thus, every symbol of $\acute{\Sigma}$ is projected to tier τ, so C_{LHOR} is k-SL.

Now, let $S \subseteq (\acute{\Sigma}, \rtimes, \ltimes)^k$ be the set of substrings that are permitted to appear in strings of C_{LHOR}, and let $x := L^k\acute{H}L^k$. Since $x \in C_{\text{LHOR}}$, every length-k substring of $\rtimes^{k-1}x\ltimes^{k-1}$ is in S. However, observe that every length-k substring of $\rtimes^{k-1}xx\ltimes^{k-1} = \rtimes^{k-1}L^k\acute{H}L^{2k}\acute{H}L^k\ltimes^{k-1}$ is also a substring of $\rtimes^{k-1}x\ltimes^{k-1}$, and is therefore also in S. Thus, we have deduced that $xx \in C_{\text{LHOR}}$, contradicting the definition of C_{LHOR}. $\square$

Proposition 19. C_{LHOR} *is 2-I-2-TSL.*

Proof. Let ψ be the homomorphic tier projection given by $\{H, \acute{H}, \acute{L}\}$, and let τ be defined by

$$\tau(x) := \begin{cases} \psi(x)L, & x \in \Sigma^*L \\ \psi(x), & \text{otherwise.} \end{cases}$$

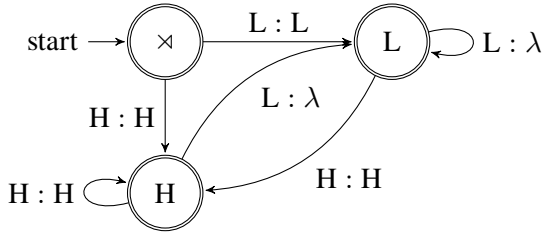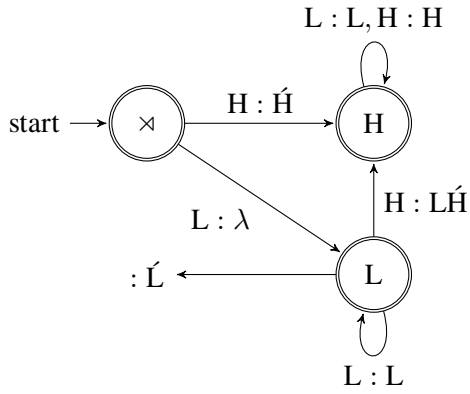In other words, τ is the same as ψ, except the last symbol of the input is always projected. It is easy

Figure 1: A 2-I-2-TISL SFST for LHOR (top) and a 2-ISL SFST for its tier projection (bottom).

to see that τ is 2-ISL. Now, observe that C_{LHOR} is 2-I-2-TSL on tier τ with permissible substrings $\rtimes\acute{H}, \rtimes\acute{L}, HH, HL, H\ltimes, \acute{H}H, \acute{H}L, \acute{H}\ltimes, \acute{L}\ltimes, L\ltimes$, and $\rtimes\ltimes$. □

As these examples illustrate, the I-TISL functions and I-TSL languages form the current subregular complexity bounds for attested subsequential stress systems and their associated stress constraints (Baek, 2018; Hao and Andersson, 2019).[2] These results extend those of Heinz (2009), Rogers et al. (2013), and Heinz (2014), who observed that stress constraints belong to restrictive subclasses of the regular languages. Other ways of refining the subregular hierarchy for stress have been proposed; Rogers and Lambert (2019), for example, define the *strictly piecewise local* and the *piecewise locally testable* language classes. The remainder of this paper will seek to compare metrical grid theory against the benchmarks I-TISL and I-TSL benchmarks.

4 Metrical Grid Theory

Treatments of stress in phonological theory are typically based on the intuition that phonemes are organized into hierarchical structures, each
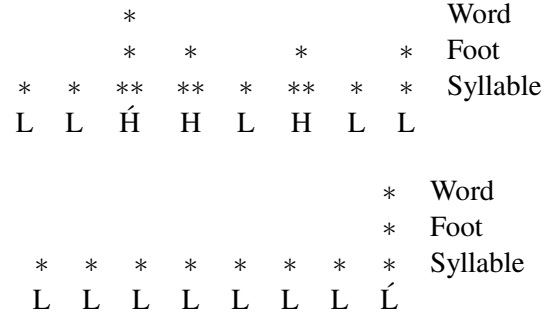

Figure 2: Sample metrical grids for the LHOR system.

level of which imposes prominence relations on its elements. While current approaches in Optimality Theory (OT) use constraints on the shapes of prosodic units and the prominence relations they impose (Prince and Smolensky, 1993, 2004; McCarthy and Prince, 1986, 1996, 1993), *metrical theory* has provided several frameworks for understanding stress outside of OT.[3] This section reviews and formalizes *metrical grid theory* (MGT, Prince, 1983; Halle and Vergnaud, 1987; Idsardi, 1992; Halle and Idsardi, 1995; Hayes, 1995), a classic example of such a framework.

According to MGT, prosodic relations are represented using diagrams like the ones that appear in Figure 2. Each syllable is associated with a continuous stack of asterisks. The height of each stack represents the prosodic prominence of its associated syllable, with the tallest stack marking primary stress and the second-tallest stack(s) marking secondary stress. Each layer of asterisks represents a level of the prosodic hierarchy: the bottom asterisks, the *syllable layer*, mark the location of each syllable; the middle asterisks, the *foot layer*, mark syllables that are prominent within their respective feet; and the top asterisk, the *word layer*, marks the syllable with the greatest prominence in the word.

The placement of asterisks within the diagram is determined as follows. In the syllable layer, all light syllables receive an asterisk, while all heavy syllables receive two asterisks ($**$). Thus, the syllable layer serves to record which syllables are heavy and which are light. In the foot layer, asterisks are placed by applying one or more of the following rules.

- **Quantity Sensitivity** (QS)**:** Place an asterisk directly above each $**$ in the syllable layer.

---

[2]Hao and Andersson (2019) and Koser and Jardine (To appear) show that some stress systems are not subsequential; I do not consider such systems in this paper.

[3]See Kager (1995) for a survey overview of various approaches in metrical theory.

- **Perfect Grid:** Place an asterisk in every second position, starting from the first (PG(odd)) or the second (PG(even)) position.

- **End Rule:** Place an asterisk in the first (ER(foot, $\lhd$)) or last (ER(foot, $\rhd$)) position.

In Figure 2, for example, foot-level asterisks are assigned according to QS and ER(foot, $\rhd$). Applying both rules means that an asterisk is added to a position if and only if *either* QS *or* ER(foot, $\rhd$) adds an asterisk to that position. Finally, the single word-layer asterisk is assigned according to the following End Rule.

- **End Rule:** Place an asterisk directly above the first (ER(word, $\lhd$)) or the last (ER(word, $\rhd$)) asterisk in the foot layer.

Using these rules, LHOR is implemented in MGT as follows: the foot-level asterisks are assigned using QS and ER(foot, $\rhd$), and the word-level asterisk is assigned using ER(word, $\lhd$). In words containing a heavy syllable, such as L^2 H́HLHL^2, the leftmost asterisk on the foot level occurs directly above the leftmost H in the word. Thus, ER(word, $\lhd$) assigns primary stress to the leftmost H. In words without a heavy syllable, such as L^8, QS does not place any asterisks on the foot layer, so the leftmost asterisk of the foot layer is the single asterisk placed by ER(foot, $\rhd$). This occurs at the right word boundary, so ER(word, $\rhd$) assigns primary stress to the rightmost syllable.

4.1 Formalizing MGT

Let us now give a precise definition of the system we have informally described. To represent stacks of asterisks, I annotate alphabet symbols with a subscript indicating the number of asterisks above that symbol. Since $**$ only occurs in the syllable layer directly above an H, I do not distinguish between a single position in the grid that contains $**$ and a single position that contains only one asterisk. For example, the upper grid in Figure 2 is represented by the string $L_1^2 H_3 H_2 L_1 H_2 L_1 L_2$.

Definition 20. Let Σ be any alphabet, and for every $\sigma \in \Sigma$, let $\sigma_1, \sigma_2, \dots$ be symbols not in $\acute{\Sigma}$. Let $\Sigma_0 := \Sigma$, and for $i \geq 0$, define the alphabet $\Sigma_i := \{\sigma_i | \sigma \in \Sigma\}$, with $\sigma_0 = \sigma$ for each $\sigma \in \Sigma$. Let $\Sigma_{\leq j} := \bigcup_{i=0}^{j} \Sigma_i$ and $\Sigma_* := \bigcup_{i=0}^{\infty} \Sigma_i$.

In this formalization, each rule must be associated with a particular level in the prosodic hierarchy. A rule associated with level i, where
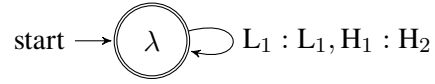

Figure 3: A homomorphic SFST implementing QS.

the syllable layer is level 1, takes as input a grid whose tallest stack of asterisks is at most i levels tall, and increments the height of stacks ending at the previous level by one.

Definition 21. For $i > 0$, a *level-i rule* is a same-length subsequential function $\rho : \Sigma_{\leq i}^* \to \Sigma_{\leq i}^*$ such that for each $x \in \Sigma_{\leq i}^*$ and for each position j,

- if $x[j] = \sigma_{i-1}$ for some $\sigma \in \Sigma$, then either $\rho(x)[j] = \sigma_{i-1}$ or $\rho(x)[j] = \sigma_i$;

- otherwise, $\rho(x)[j] = x[j]$.

Example 22. Figure 3 shows an SFST implementing QS as a level-2 rule. Since L_1 represents a column with a single asterisk and H_1 represents a column with $**$, this SFST simply changes all H_1s to H_2. ER$(i, \lhd)$ is represented by the following level-i rule:

$$\text{ER}_i^{\lhd}(x) = \begin{cases} y\sigma_i z, & x = y\sigma_{i-1}z \text{ and } y \in \Sigma_{\leq i-2}^* \\ x, & \text{otherwise.} \end{cases}$$

ER$(i, \lhd)$ places an asterisk above the leftmost asterisk on level $i - 1$. Symbols in $\Sigma_i \cup \Sigma_{i-1}$ represent syllables with an asterisk on level $i - 1$; symbols in Σ_i represent syllables with an asterisk on both level $i - 1$ and level i. If the first symbol of x in $\Sigma_i \cup \Sigma_{i-1}$ is of the form $\sigma_{i-1} \in \Sigma_{i-1}$, then this symbol is incremented to σ_i. If the first symbol of x in $\Sigma_i \cup \Sigma_{i-1}$ is of the form $\sigma_i \in \Sigma_i$, then this symbol is left unchanged: ER$(i, \lhd)$ is still understood to add an asterisk on the ith level, but an asterisk has already been added there by another rule. If x does not contain any symbols of $\Sigma_i \cup \Sigma_{i-1}$, then ER$(i, \lhd)$ does not add any asterisks. Observe that ER$_i^{\lhd}$ is 2-TISL on tier $\Sigma_i \cup \Sigma_{i-1}$.

The mapping of input words to their metrical-grid representations is simply the composition of a sequence of rules. Since rules can only place asterisks on top of existing asterisks from the previous layer, the rules in the sequence are required to be monotonically increasing in their associated level of the hierarchy.[4]

---

[4] This requirement is known in the phonological literature as the *continuous column constraint* (Hayes, 1995).

Definition 23. For $i > 0$, an *i-level metrical grid* is a function $\rho : \Sigma^* \to \Sigma^*_{\leq i}$ such that

$$\rho = \rho_n \circ \rho_{n-1} \circ \cdots \circ \rho_0$$

for some $n > 0$, where

- $\rho_0 : \Sigma^* \to \Sigma^*_1$ is the homomorphism given by $\rho(\sigma) = \sigma_1$ for all $\sigma \in \Sigma$;

- ρ_n is a level-i rule; and

- for all j, if ρ_j is a level-k rule, then ρ_{j+1} is either a level-k rule or a level-$(k+1)$ rule.

From an i-level metrical grid, we recover the stress system described by the grid by assuming that asterisks on level i represent primary stress.

Definition 24. Let ρ be an i-level metrical grid. The *stress system induced by ρ* is the stress system $s_\rho := s_i \circ \rho$, where $s_i : \Sigma^*_{\leq i} \to \acute{\Sigma}^*$ is the homomorphism given by

$$s_i(\sigma_j) := \begin{cases} \acute{\sigma}, & j = i \\ \sigma, & j < i. \end{cases}$$

5 Expressive Power of MGT

Trivially, the version of MGT formalized in Subsection 4.1 can express any subsequential stress system s: since level-i rules are allowed to be arbitrary subsequential functions, it suffices to construct a grid consisting of a level-2 rule that places an asterisk above the syllable assigned primary stress by s. In this section, I show that MGT is strictly more expressive than the I-TISL functions. The example that separates MGT from the I-TISL functions is motivated by Hao and Andersson's (2019) formalization of *Dybo's Rule* (Dybo, 1977), a description of unbounded stress in Abkhaz. I review Hao and Andersson's implementation of Dybo's Rule both as a stress system and as a 3-level grid in Subsection 5.1. In Subsection 5.2, I show that a slight modification of Hao and Andersson's stress system is in fact not I-TISL, even though the ability of MGT and I-TSL languages to describe the system is not affected by the change in representation, as will be shown in Subsection 5.3.

5.1 Dybo's Rule

In Abkhaz, syllables are lexically marked as being *dominant* or *recessive*. Dybo's Rule is an LHOR stress system in which dominant syllables not followed by other dominant syllables are considered to be heavy, and all other syllables are considered to be light. This is illustrated by the following examples, where dominant syllables are underlined.

(25)   Dybo's Rule in Abkhaz (Spruit, 1986)
   a. [dət͡ʃʰala'wama] 'Does (s)he usually go?'
   b. [a'ʁʷak'ʲaməsa] '(the) poniard'
   c. [apʰa'ra] 'to pleat'
   d. [maa'k'ə] 'one handle'

The dominant syllables [wa] in (25a), [ʁʷa] and [ma] in (25b), and [ra] in (25c) are heavy, since they are not followed by another dominant syllable. In (25a) and (25c), the sole heavy syllable receives primary stress. In (25b), the first of the two heavy syllables receives primary stress. (25d) does not have any heavy syllables, so the last syllable receives primary stress by default. Hao and Andersson (2019) represent Dybo's Rule using the following stress system.

Definition 26. Let $\Sigma := \{D, R\}$. The *two-letter Dybo's Rule* is the stress system $\alpha : \Sigma^* \to \acute{\Sigma}^*$ defined as follows. For $u \in R^*D^*$, $v \in \Sigma$, and $w \in \Sigma^*$,

$$\alpha(uvw) := \begin{cases} u\acute{v}w, & v = D \text{ and } w \notin D\Sigma^* \\ uw\acute{v}, & uvw \in R^*. \end{cases}$$

In the two-letter Dybo's Rule, dominant syllables are represented by D and recessive syllables are represented by R. The first D not followed by an R receives primary stress. This stress system turns out to be an I-TISL function.

Proposition 27 (Hao and Andersson, 2019). *The two-letter Dybo's Rule is 2-I-3-TISL.*

Hao and Andersson implement this system in MGT using a 3-level grid of the form $\mathrm{ER}_3^{\lhd} \circ \rho_1 \circ \rho_0$, where ρ_0 is as defined in Definition 23 and ρ_1 is given by the 2-ISL SFST shown in the right panel of Figure 4. Following the MGT analysis of LHOR stress, ρ_1 serves to mark all heavy syllables, as well as the last syllable, with an asterisk on level 2. Thus, ρ_1 places an asterisk above all Ds followed by an R, along with the last syllable.
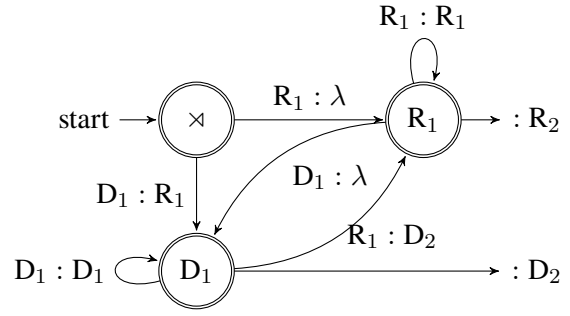
Figure 4: **Left:** Sample metrical grids for the two-letter Dybo's Rule. **Right:** 2-ISL SFST implementing the level-2 rule for the two-letter Dybo's Rule.

5.2 MGT vs. I-TISL Functions

Let us now introduce the following variant of the two-letter Dybo's Rule.

Definition 28. Let $\Sigma := \{D, E, R\}$. The *three-letter Dybo's Rule* is the stress system $\delta : \Sigma^* \to \acute{\Sigma}^*$ defined as follows. For $u \in R^*\{D, E\}^*$, $v \in \Sigma$, and $w \in \Sigma^*$,

$$\delta(uvw) := \begin{cases} u\acute{v}w, & v \neq R \text{ and } w \notin \{D, E\}\Sigma^* \\ uw\acute{v}, & uvw \in R^+. \end{cases}$$

The three-letter Dybo's Rule is exactly like the two-letter Dybo's Rule, except that there are two alphabet symbols representing dominant syllables: D and E. The MGT analysis of the two-letter Dybo's Rule can be easily adapted to the three-letter Dybo's Rule just by identifying E with D. The I-TISL implementation of the two-letter Dybo's Rule, however, cannot be applied to the three-letter Dybo's Rule.

Theorem 29. *The three-letter Dybo's Rule is not i-I-j-TISL on tier τ for any i, j, or τ.*

Proof. Suppose δ is i-I-j-TISL on tier τ. Observe that

$$\delta^{\to}_{E^iD}(R) = \acute{D}R \qquad \delta^{\to}_{E^iE}(R) = \acute{E}R$$
$$\delta^{\to}_{E^iR}(R) = R \qquad \delta^{\to}_{E^i}(R) = \acute{E}R,$$

thus $\tau^{\leftarrow}(E^iD)$, $\tau^{\leftarrow}(E^iR)$, and $\tau^{\leftarrow}(E^i)$ must all be distinct. Let $t := \tau^{\leftarrow}(E^i)$, and for $\sigma \in \Sigma$, let t_σ be such that $\tau^{\leftarrow}(E^i\sigma) = tt_\sigma$. Clearly, $t_\sigma \neq \lambda$ for every σ.

Let q_0 be the start state of the minimal SFST T for τ, let q be the state of T corresponding to $\tau^{\to}_{E^i}$, let $\to$ be the transition function of T, and let r, y, and z be such that

$$q_0 \xrightarrow{E^i:y} q \xrightarrow{DE^i:z} r.$$

Since τ is i-ISL and

$$E^i[: 1 - i] = E^iDE^i[: 1 - i] = E^{i-1},$$

we must have $\tau^{\to}_{E^iDE^i} = \tau^{\to}_{E^i}$, thus $r = q$. It follows that for every $k \geq 0$, $\tau^{\leftarrow}(E^i(DE^i)^k) = yz^k$. Since $t_E \preceq z$, it must be the case that $|yz^k| \geq k$.

Now, observe that

$$\tau^{\leftarrow}(DRE^i(DE^i)^j)[: 1 - j]$$
$$= \tau^{\leftarrow}(DRE^i)z^j[: 1 - j]$$
$$= yz^j[: 1 - j]$$
$$= \tau^{\leftarrow}(E^i(DE^i)^j)[: 1 - j].$$

Therefore, $\tau^{\to}_{DRE^i(DE^i)^j} = \tau^{\to}_{E^i(DE^i)^j}$. However, this contradicts the fact that

$$\tau^{\to}_{DRE^i(DE^i)^j}(R) = R \neq \acute{E}R = \tau^{\to}_{E^i(DE^i)^j}(R),$$

so we conclude that δ is not i-I-j-TISL on tier τ for any i, j, or τ. $\square$

The 2-I-3-TISL SFST given by Hao and Andersson (2019) for the two-letter Dybo's Rule projects DR sequences to the tier. When the SFST encounters a contiguous block of Ds, it must delay its output by one time step, as shown below, because it is unknown whether or not the current D should be assigned stress. Stress is not assigned until the SFST encounters an R or the end of the input string has been reached.

$$\rtimes\rtimes \xrightarrow{D:\lambda} \rtimes D \xrightarrow{D:D} \ldots \xrightarrow{D:D} \rtimes D \xrightarrow{R:\acute{D}R} DR$$

Once the tier contains a full DR sequence, the SFST knows that stress has already been assigned, and therefore does not assign stress for the remainder of its computation.

With the three-letter Dybo's Rule, the state needs to record the identity of the most recent input

symbol in order to delay the output by one time step. The only way to do this with an I-TISL SFST is to project the most recent input symbol to the tier.

$$\rtimes\rtimes \xrightarrow{\text{D:}\lambda} \rtimes\text{D} \xrightarrow{\text{E:D}} \text{DE} \xrightarrow{\text{D:E}} \text{ED} \rightarrow \dots$$

Since an ISL tier projection cannot distinguish between the first block of dominant syllables in its input and subsequent blocks of dominant syllables, the schema shown above requires *every* block of dominant syllables to be projected to the tier. These syllable blocks overflow the memory provided by the tier, thus preventing it from recording whether or not stress has already been assigned.

5.3 MGT vs. I-TSL Languages

Despite the fact that the three-letter Dybo's Rule is not I-TISL, the stress constraint it induces is I-TSL.

Proposition 30. C_δ *is* 2-*I*-3-*TSL*.

Proof. Let τ be the 2-ISL tier projection that projects

- all instances of D́, É, and Ŕ;

- all instances of DR, DŔ, ER, EŔ, D́R, D́Ŕ, ÉR, and ÉŔ; and

- the last symbol of the input.

Now, observe that C_δ is 2-I-3-TSL on tier τ, with the following permissible substrings: $\rtimes\rtimes\acute{\sigma}$, $\rtimes\rtimes\acute{\text{R}}$, $\rtimes\rtimes\ltimes$, $\rtimes\acute{\sigma}\text{R}$, $\rtimes\acute{\sigma}\ltimes$, $\rtimes\acute{\text{R}}\ltimes$, $\rtimes\ltimes\ltimes$, $\sigma\text{R}\gamma$, $\sigma\text{R}\ltimes$, $\text{R}\sigma\text{R}$, $\text{R}\ltimes\ltimes$, $\acute{\sigma}\text{R}\gamma$, $\acute{\sigma}\text{R}\ltimes$, $\acute{\sigma}\ltimes\ltimes$, and $\acute{\text{R}}\ltimes\ltimes$, where $\sigma, \gamma \in \{\text{D}, \text{E}\}$. $\quad\square$

The tier projection described here is similar to the tier projection used for Hao and Andersson's (2019) 2-I-3-TISL implementation of α. Like the 2-I-2-TSL grammar for C_{LHOR}, the 2-I-3-TSL grammar for C_δ projects all heavy syllables and stressed syllables to the tier, along with the last syllable of the input. Unlike the grammar for C_{LHOR}, the grammar for C_δ also projects recessive syllables following dominant syllables. This allows the grammar to ensure that all stressed dominant syllables are dominant: they must be immediately followewd by either R or $\ltimes$.

Because neither an ITSL grammar nor a metrical grid needs to produce the surface form as output, the problem of using the tier to delay computation does not arise for the ITSL implementation of C_δ or for the MGT analysis of δ. While there is still a discrepancy between layer 2 of the MGT analysis and the tier projection used for C_δ, I conjecture based on this observation that MGT describes ITSL decision problems.

Conjecture 31. *Let* $\rho = \text{ER}_3^{\lhd} \circ \rho_1 \circ \rho_0$ *be a 3-level metrical grid. If* ρ_1 *is ISL, then* C_{s_ρ} *is I-TSL.*

6 Conclusion

In comparing the I-TISL implementation of α with the MGT analysis, Hao and Andersson (2019) express the intuition that generalized tier projections and MGT are similar in that both systems use intermediate representations in order to compute stress. The analysis of Section 5 has revealed that this similarity is superficial because the computations carried out by I-TISL functions, I-TSL languages, and metrical grids are fundamentally different from one another. The most prominent of the differences discussed here is that systems implementing the transduction problem need to transfer a substantial amount of information about the underlying form to the surface form, while systems implementing the decision problem only need to retain enough information to distinguish a grammatical string from an ungrammatical one. Thus, the transduction problem may be viewed as conceptually more difficult than the decision problem.[5] While metrical grids compute transductions, their memory capabilities are enhanced by the fact that rule composition allows state information to be encoded in intermediate layers. Conjecture 31 suggests that this enhanced memory may be sufficient for MGT to bridge the gap between the transduction problem and the decision problem.

In conclusion, the comparison of generalized tier projection with metrical grids provides an instructive example of an analytical tool—intermediate representations of prominence relations—that behaves differently depending on the formalism in which it is instantiated. This approach could potentially offer a way to compare different theoretical frameworks in terms of how they accommodate superfically similar proposals. I leave the exploration of such ideas to future work.

---

[5]This asymmetry mirrors the relationship between *search problems* and *decision problems* in computational complexity theory (see Arora and Barak, 2009, pp. 54–55 for an overview). The conjecture that NP $\supsetneq$ P captures the intuition that the search problem is the more difficult one.

References

Sanjeev Arora and Boaz Barak. 2009. *Computational Complexity: A Modern Approach*. Cambridge University Press, Cambridge, United Kingdom.

Emmon Bach. 1975. Long Vowels and Stress in Kwakiutl. *Texas Linguistic Forum*, 2:9–19.

Hyunah Baek. 2018. Computational representation of unbounded stress: Tiers with structural features. In *Proceedings of CLS 53 (2017)*, volume 53, pages 13–24, Chicago, IL. Chicago Linguistic Society.

Franz Boas, Helene Boas Yampolsky, and Zellig S. Harris. 1947. Kwakiutl Grammar with a Glossary of the Suffixes. *Transactions of the American Philosophical Society*, 37(3):203–377.

Phillip Burness and Kevin McMullin. 2019. Efficient learning of Output Tier-based Strictly 2-Local functions. In *Proceedings of the 16th Meeting on the Mathematics of Language*, pages 78–90, Toronto, Canada. Association for Computational Linguistics.

Jane Chandlee. 2014. *Strictly Local Phonological Processes*. PhD Dissertation, University of Delaware, Newark, DE.

Jane Chandlee, Rémi Eyraud, and Jeffrey Heinz. 2015. Output Strictly Local Functions. In *Proceedings of the 14th Meeting on the Mathematics of Language*, pages 112–125, Chicago, IL. Association for Computational Linguistics.

Noam Chomsky and Morris Halle. 1968. *The Sound Pattern of English*, 1 edition. Harper & Row, New York, NY.

Vladimir Antonovič Dybo. 1977. Западнокавказская акцентная система и проблема ее происхождения. In *Конференция «Ностратические языки и ностратическое языкознание»: Тезисы докладов*, pages 41–45, Moscow, Soviet Union. USSR Academy of Sciences.

John Anton Goldsmith. 1976. *Autosegmental Phonology*. PhD Dissertation, Massachusetts Institute of Technology, Cambridge, MA.

Thomas Graf and Connor Mayer. 2018. Sanskrit n-Retroflexion is Input-Output Tier-Based Strictly Local. In *Proceedings of the Fifteenth Workshop on Computational Research in Phonetics, Phonology, and Morphology*, pages 151–160, Brussels, Belgium. Association for Computational Linguistics.

Morris Halle and William J. Idsardi. 1995. General Properties of Stress and Metrical Structure. In John A. Goldsmith, editor, *The Handbook of Phonological Theory*, 1 edition, Blackwell Handbooks in Linguistics, pages 403–443. Wiley-Blackwell, Cambridge, MA.

Morris Halle and Jean-Roger Vergnaud. 1987. *An Essay On Stress*. Number 15 in Current Studies in Linguistics. MIT Press, Cambridge, MA.

Yiding Hao and Samuel Andersson. 2019. Unbounded Stress in Subregular Phonology. In *Proceedings of the 16th SIGMORPHON Workshop on Computational Research in Phonetics, Phonology, and Morphology*, pages 135–143, Florence, Italy. Association for Computational Linguistics.

Bruce Hayes. 1995. *Metrical Stress Theory: Principles and Case Studies*. University of Chicago Press, Chicago, IL.

Jeffrey Heinz. 2009. On the role of locality in learning stress patterns. *Phonology*, 26(2):303–351.

Jeffrey Heinz. 2014. Culminativity times harmony equals unbounded stress. In Harry van der Hulst, editor, *Word Stress: Theoretical and Typological Issues*, pages 255–275. Cambridge University Press, Cambridge, United Kingdom.

Jeffrey Heinz. 2018. The computational nature of phonological generalizations. In Larry M. Hyman and Frans Plank, editors, *Phonological Typology*, number 23 in Phonology and Phonetics, pages 126–195. De Gruyter Mouton, Berlin, Germany.

Jeffrey Heinz, Chetan Rawal, and Herbert G. Tanner. 2011. Tier-based Strictly Local Constraints for Phonology. In *Proceedings of the 49th Annual Meeting of the Association for Computational Linguistics: Human Language Technologies*, pages 58–64, Portland, OR. Association for Computational Linguistics.

William James Idsardi. 1992. *The Computation of Prosody*. PhD Dissertation, Massachusetts Institute of Technology, Cambridge, MA.

René Kager. 1995. The Metrical Theory of Word Stress. In John A. Goldsmith, editor, *The Handbook of Phonological Theory*, 1 edition, Blackwell Handbooks in Linguistics, pages 367–402. Wiley-Blackwell, Cambridge, MA.

Nate Koser and Adam Jardine. To appear. Stress assignment and subsequentiality. In *Proceedings of the 2019 Annual Meeting on Phonology*, Proceedings of the Annual Meetings on Phonology, Stony Brook, NY. Linguistic Society of America.

Connor Mayer and Travis Major. 2018. A Challenge for Tier-Based Strict Locality from Uyghur Backness Harmony. In *Formal Grammar 2018, 23rd International Conference, FG 2018, Sofia, Bulgaria, August 11-12, 2018, Proceedings*, volume 10950 of *Lecture Notes in Computer Science*, pages 62–83, Berlin, Germany. Springer Berlin Heidelberg.

John J. McCarthy and Alan Prince. 1986. Prosodic Morphology. Technical Report 32, Rutgers Center for Cognitive Science, New Brunswick, NJ.

John J. McCarthy and Alan Prince. 1993. Generalized alignment. In Geert Booij and Jaap Van Marle, editors, *Yearbook of Morphology 1993*, Yearbook of Morphology, pages 79–153. Springer Netherlands, Dordrecht, Netherlands.

John J. McCarthy and Alan Prince. 1996. Prosodic Morphology 1986. Submission 13, University of Massachusetts Amherst Department of Linguistics, Amherst, MA.

Adam McCollum, Eric Baković, Anna Mai, and Eric Meinhardt. 2018. The expressivity of segmental phonology and the definition of weak determinism. *LingBuzz*, lingbuzz/004197.

Alan Prince and Paul Smolensky. 1993. Optimality Theory: Constraint Interaction in Generative Grammar. Technical Report 2, Rutgers University, New Brunswick, NJ.

Alan Prince and Paul Smolensky. 2004. *Optimality Theory: Constraint Interaction in Generative Grammar*. Blackwell Publishing, Malden, MA.

Alan S. Prince. 1983. Relating to the Grid. *Linguistic Inquiry*, 14(1):19–100.

George N. Raney. 1958. Sequential Functions. *Journal of the Association for Computing Machinery*, 5(2):177–180.

James Rogers, Jeffrey Heinz, Margaret Fero, Jeremy Hurst, Dakotah Lambert, and Sean Wibel. 2013. Cognitive and Sub-regular Complexity. In *Formal Grammar*, volume 8036 of *Lecture Notes in Computer Science*, pages 90–108, Berlin, Germany. Springer Berlin Heidelberg.

James Rogers and Dakotah Lambert. 2019. Some classes of sets of structures definable without quantifiers. In *Proceedings of the 16th Meeting on the Mathematics of Language*, pages 63–77, Toronto, Canada. Association for Computational Linguistics.

Jacques Sakarovitch. 2009. *Elements of Automata Theory*. Cambridge University Press, Cambridge, United Kingdom.

Arie Spruit. 1986. *Abkhaz Studies*. PhD Dissertation, Leiden University, Leiden, Netherlands.

Odile Vaysse. 1986. Addition molle et fonctions p-locales. *Semigroup Forum*, 34(1):157–175.