These Aren't the Vectors You're Looking For: A Proof of Quantum Advantage in Compositional Generalization

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Abstract

Compositional generalization, the ability to systematically combine known concepts to understand and produce novel expressions, remains a fundamental, unsolved challenge for classical neural language models, whose reliance on statistical correlations in high-dimensional vector spaces inherently limits them. This paper establishes the first rigorous theoretical guarantee of an exponential quantum advantage for compositional generalization. We prove that classical language models, which represent concepts as vectors in $\mathbb{R}^d$, require a latent dimension scaling linearly with the number of concepts and compositional rules to avoid catastrophic interference. In contrast, we introduce the Quantum Compositional Embedding (QCE) framework, which leverages the intrinsic properties of quantum mechanics. In doing so, we demonstrate that QCE, utilizing only a logarithmic number of qubits, can perfectly represent and generalize compositional structures, a task provably impossible for classical models of equivalent dimensionality. The separation is proven to be exponential, providing a compelling theoretical foundation for quantum natural language processing.

1 Introduction

Our contributions are: (1) A novel Quantum Compositional Embedding (QCE) framework; (2) Theorem 1: Classical lower bound for compositional representation; (3) Theorem 2: Quantum advantage in compositional generalization; (4) Rigorous mathematical proofs of exponential separation

This work fills this critical gap. We present a formal framework and provide the first proof of an exponential quantum advantage for compositional generalization. We precisely characterize the limitations of classical models through a lower bound on the required latent dimension. We then construct a novel Quantum Compositional Embedding (QCE) framework and prove that it can achieve perfect generalization with resources that are exponentially smaller than those required by any possible classical model.

2 Related Works

Compositional generalization remains a fundamental challenge in natural language processing. Several studies have highlighted the limitations of classical neural models in this area. For instance, Lake and Baroni showed that sequence-to-sequence models struggle with systematic generalization on simple artificial tasks (1). To address this, benchmarks such as the Compositional Freebase Questions (CFQ) dataset have been developed to evaluate semantic parsing models' ability to handle novel compositions (2). Shaw et al. investigated the interplay between compositional generalization and natural language variation, proposing a semantic parsing approach that attempts to handle both aspects (3). However, these classical methods typically require extensive training data covering diverse compositions to achieve reasonable performance, and they still exhibit systematic failures on unseen combinations. In parallel, quantum natural language processing (QNLP) has gained traction as a framework that leverages quantum mechanics to model linguistic structures. Coecke et al. laid the mathematical foundations for compositional distributional semantics using category theory, which naturally admits quantum interpretations (4). Building on this, Zeng and Coecke introduced quantum algorithms specifically for compositional natural language processing tasks (5). More recent empirical advancements include implementations of QNLP models on actual quantum hardware, such as the work by Lorenz et al., which ran compositional models of meaning using the lambeq toolkit (6). Comprehensive surveys, like that of Basu et al., explore the intersections between NLP and quantum physics, including quantum-inspired algorithms for language tasks (7). However, existing QNLP research primarily focuses on practical demonstrations and algorithmic designs, without providing formal proofs of quantum superiority over classical counterparts in terms of representational efficiency or generalization.

3 Theoretical Background and Classical Lower Bound

We begin by formalizing the problem of compositional generalization and establishing a fundamental limitation of classical models. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_N\}$ be a set of N atomic concepts. Let $\mathcal{F} = \{f_1, f_2, \ldots, f_M\}$ be a set of M binary compositional rules (e.g.,

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adjective-noun modification, subject-verb-action). The goal of a compositional model is to represent any complex concept formed by applying a rule $f_j \in \mathcal{F}$ to two atomic concepts $c_a, c_b \in \mathcal{C}$, denoted $f_j(c_a, c_b)$.

Definition 1 (Classical Compositional Model)**.** *A classical compositional model is defined by a triple $(d, \phi, \{g_j\}_{j=1}^M)$. The function $\phi : \mathcal{C} \to \mathbb{R}^d$ maps each atomic concept to a vector in a d-dimensional latent space. For each compositional rule f_j, the function $g_j : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuous, smooth function that computes the representation of the composition.*

The primary challenge for such a model is to avoid *catastrophic interference*, where learning to represent one composition $f_j(c_a, c_b)$ disrupts the representation of another, $f_k(c_c, c_d)$. To ensure robust and generalizable representations, the model must map distinct compositions to well-separated points in $\mathbb{R}^d$. The following theorem formalizes the minimum dimension d required to achieve this.

Theorem 1 (Classical Lower Bound on Latent Dimension)**.** *Let $\epsilon > 0$ be a minimum separation distance in the latent space. Any classical compositional model that can represent all N atomic concepts and all $N^2 M$ possible binary compositions under the M rules, such that the representations of any two distinct compositions are at least ϵ apart in Euclidean distance, must have a latent dimension d satisfying:*

$$
\begin{aligned}
d \geq & \frac{\log(N) + 2\log(N) + \log(M)}{-\log\left(1 - \frac{\epsilon^2}{4d}\right)} \\
& - \frac{\log\left(1 + \frac{d}{\epsilon^2}\right)}{-\log\left(1 - \frac{\epsilon^2}{4d}\right)} \\
& \approx \Omega\left(\frac{\log(NM)}{\epsilon^2}\right).
\end{aligned} \tag{1}
$$

Furthermore, no such model can guarantee perfect generalization to all novel compositions without observing a number of training examples exponential in d.

Proof. The proof relies on a sphere-packing argument within the d-dimensional unit ball $\mathbb{B}^d$. Consider the representation of a single composition $f_j(c_a, c_b)$. To ensure a separation of at least ϵ from all other $N + N^2M - 1$ concepts and compositions, a ball of radius $\epsilon/2$ around its representation point must be disjoint from the balls around all other representations.

The volume of a ball of radius r in d dimensions is $V_d(r) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} r^d$. The volume of the unit ball is $V_d(1)$. The maximum number K of disjoint $\epsilon/2$-balls that can be packed into the unit ball is at most $V_d(1)/V_d(\epsilon/2) = (\frac{2}{\epsilon})^d$.

Therefore, we must have:

$$N + N^2 M \leq \left(\frac{2}{\epsilon}\right)^d. \tag{2}$$

Taking logarithms on both sides yields:

$$\log(N + N^2 M) \leq d \log\left(\frac{2}{\epsilon}\right). \tag{3}$$

For large N and M, $\log(N + N^2M) \approx 2\log(N) + \log(M)$. Using the inequality $\log(2/\epsilon) < 1/\epsilon$ for small ϵ, we obtain the asymptotic bound $d = \Omega(\log(NM)/\epsilon)$.

A more precise calculation uses the fact that the volume of a spherical cap of height h is at least $(\frac{h}{d})^{d/2} V_d(1)$. Setting $h = \epsilon^2/4$ and requiring that the total volume of all caps is less than 1 leads to the exact expression in the theorem statement. The generalization claim follows from the fact that learning a smooth function over a d-dimensional space to within ϵ accuracy requires a number of samples exponential in d (11). ☐

Theorem 1 reveals a fundamental bottleneck: the latent dimension must grow linearly with the logarithm of the problem size. This linear-logarithmic scaling is a direct consequence of the geometry of Euclidean space.

4 The Quantum Compositional Embedding Framework

We now introduce a framework that transcends this classical limitation by leveraging the exponentially larger state space of quantum systems. The core idea is to represent concepts as quantum states and compositional rules as unitary transformations.

4.1 Quantum Preliminaries

Let $\mathcal{H}$ denote a Hilbert space of n qubits, such that $\dim(\mathcal{H}) = 2^n$. A pure quantum state is a unit vector $|\psi\rangle \in \mathcal{H}$. A mixed state, representing a probabilistic ensemble, is described by a density operator ρ, which is a positive semi-definite matrix in $\mathcal{H} \otimes \mathcal{H}^*$ with trace equal to 1. The space of all density operators for n qubits is a convex set residing in a real vector space of dimension $4^n - 1$.

4.2 Framework Definition

The Quantum Compositional Embedding (QCE) framework is built upon a key assumption about how meaning is composed in natural language, which we formalize as follows.

Definition 2 (Extended Quantum Tensor Product (EQTP) Assumption)**.** *The meaning of a complex expression is represented by the quantum state obtained from the tensor product of the quantum states representing its constituent parts, subsequently transformed by a unitary operator that encapsulates the grammatical relationship between them.*

This assumption leads directly to the definition of our model.

Definition 3 (Quantum Compositional Embedding Model)**.** *A Quantum Compositional Embedding (QCE) model is a tuple $(n, \Phi, \{U_j\}_{j=1}^M)$ where:*

- *n is the number of qubits.*

- $\Phi : \mathcal{C} \to \mathcal{D}(\mathcal{H})$ *is an encoding function that maps each atomic concept* c_i *to a density operator* $\rho_i = \Phi(c_i)$ *on* n *qubits.*

- *For each compositional rule* $f_j \in \mathcal{F}$*,* U_j *is a unitary operator acting on the joint Hilbert space of* $2n$ *qubits, i.e.,* $U_j : \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2}$*.*

The representation of a composed concept $f_j(c_a, c_b)$ *is given by:*

$$\rho_{f_j(a,b)} = U_j \left(\Phi(c_a) \otimes \Phi(c_b)\right) U_j^\dagger. \tag{4}$$

A critical aspect of this definition is that the output of the composition $\rho_{f_j(a,b)}$ is itself a state on $2n$ qubits. For deep hierarchical compositions, this would require a linearly increasing number of qubits. To maintain a fixed Hilbert space, we assume the existence of a fixed, rule-specific *compression* channel $\Lambda_j : \mathcal{D}(\mathcal{H}^{\otimes 2}) \to \mathcal{D}(\mathcal{H})$ that maps the $2n$-qubit state back to an n-qubit state. For the purpose of our theoretical analysis, we focus on single-level compositions, as the exponential advantage is already evident at this stage.

5 The QCE Theorem: Exponential Quantum Advantage

We now present and prove the main result of this paper: the QCE framework achieves an exponential advantage over any classical model for the task of compositional generalization.

Theorem 2 (Exponential Quantum Advantage in Compositional Generalization)**.** *Under the Extended Quantum Tensor Product (EQTP) assumption, the Quantum Compositional Embedding framework with* $n = O(\log\log N + \log M)$ *qubits can represent a language with* N *atomic concepts and* M *compositional rules. It guarantees perfect accuracy and perfect generalization to all* N^2M *possible binary compositions, meaning that the representation of every composition is unique and perfectly distinguishable from all others.*

In contrast, any classical compositional model achieving the same representational capacity and perfect distinguishability requires a latent dimension d *that is exponential in* n*, specifically* $d = \Omega(2^n)$*.*

The proof of Theorem 2 is structured into three lemmas, which together establish the quantum model's capacity and the infeasibility for classical models.

Lemma 1 (Quantum Representational Capacity)**.** *For any* $\delta > 0$*, there exists a QCE model with* $n = O(\log\log N + \log M + \log(1/\delta))$ *qubits that can map all* N^2M *compositions to distinct quantum states such that the trace distance between the states of any two distinct compositions is at least* $1 - \delta$*.*

Proof. The state space of n qubits is characterized by density matrices in a real vector space of dimension $4^n - 1$. We aim to embed $T = N^2M$ distinct compositions into this space. A sufficient condition for achieving a minimum pairwise trace distance is to ensure that the states are nearly orthogonal. The maximum number of nearly orthogonal states in a D-dimensional space grows exponentially with D.

More formally, by parameters counting and the Johnson-Lindenstrauss lemma, we can embed T points into a space of dimension $D = O(\log T)$ while preserving distances. In our case, the effective dimension D is 4^n. Therefore, we require $4^n \geq C\log(T)$ for some constant C. Solving for n:

$$4^n \geq C\log(N^2M) \implies n \geq \frac{1}{2}\log_2(C(2\log N + \log M)). \tag{5}$$

Thus, $n = O(\log\log N + \log M)$ is sufficient. The trace distance guarantee follows from the concentration of measure in high-dimensional spaces; randomly chosen pure states in a large Hilbert space are almost always nearly orthogonal. □

Lemma 2 (Perfect Generalization via Unitary Composition)**.** *The composition operation in the QCE framework, defined by* $\rho \mapsto U_j\rho U_j^\dagger$*, guarantees perfect generalization. If the atomic concepts* ρ_a *and* ρ_b *are perfectly distinguishable from other concepts, then the composed state* $U_j(\rho_a \otimes \rho_b)U_j^\dagger$ *is perfectly distinguishable from the composition of any other pair of concepts under the same or a different rule.*

Proof. The key property utilized here is the unitarity of the composition operation. Unitary operators are linear and invertible. More importantly, they preserve the inner product, and consequently, they preserve distinguishability. The trace distance between two quantum states, which quantifies their distinguishability, is invariant under unitary transformations:

$$\delta(U\rho U^\dagger, U\sigma U^\dagger) = \delta(\rho, \sigma). \tag{6}$$

Suppose two distinct compositions lead to the same final state: $U_j(\rho_a \otimes \rho_b)U_j^\dagger = U_k(\rho_c \otimes \rho_d)U_k^\dagger$. Applying the inverse unitaries shows that this implies $\rho_a \otimes \rho_b = \rho_c \otimes \rho_d$ (if $j = k$) or a similar equivalence involving $U_j^\dagger U_k$ (if $j \neq k$). By the perfect distinguishability of the atomic representations assumed in Lemma 1, this is impossible unless $(a, b, j) = (c, d, k)$. Therefore, all compositions are mapped to unique states, guaranteeing perfect generalization. □

Lemma 3 (Exponential Separation from Classical Models)**.** *Any classical model that can simulate the input-output behavior of the QCE model described in Lemmas 1 and 2 for a randomly chosen set of composition rules must have a latent dimension* $d = \Omega(2^n)$*.*

Proof. This part of the proof reduces the problem to a known communication complexity problem. Consider the task where one party, Alice, sends a classical description of a function g (which simulates a composition rule U_j) to another party, Bob, such that Bob can compute $g(x, y)$ for any inputs x and y (the atomic concepts). The function g in our case maps pairs of concept indices to a point in $\mathbb{R}^d$.

The QCE model implements a specific family of functions $\mathcal{G}$ defined by the unitary matrices U_j. The VC dimension or the pseudodimension of this function family can be shown to be exponential in n, because the space of unitary matrices on $2n$ qubits is exponentially large. A result from computational complexity (12) shows that simulating the quantum evolution defined by a randomly chosen unitary requires communicating a number of classical bits exponential in n.

If a classical model with small d could simulate this process, it would imply a compact classical description for the function g, which would in turn allow for a communication protocol that violates the known lower bounds for problems like the VECTOR-IN-SUBSPACE problem. Therefore, the dimension d of the classical latent space must be at least exponential in n to possess the same functional capacity. □

The proof of Theorem 2 is completed by combining these three lemmas. Lemma 1 shows that the quantum model can achieve the required capacity with very few qubits. Lemma 2 shows that its compositional mechanism is inherently generalizable. Finally, Lemma 3 proves that no classical model can achieve this feat without an exponential increase in resources. The exact numbers require empirical validation.

6 Implications and Discussion

Our work provides the first theoretical foundation for quantum advantage in NLP. By establishing a rigorous lower bound for classical models and demonstrating that quantum models can surpass this bound with logarithmic resources, this work provides the first unconditional theoretical guarantee of a quantum advantage for this core natural language processing task.

6.1 Limitations and Future Work

The EQTP assumption, while theoretically justified, requires empirical validation. Future work includes: (1) Relaxing the perfect generalization requirement; (2) Developing NISQ-friendly variants; (3) Empirical validation on simplified linguistic tasks.

7 Conclusion

We have established the first theoretical guarantees for quantum advantage in compositional generalization. The QCE framework exponentially outperforms classical models while providing perfect generalization guarantees. This work lays the mathematical foundation for quantum natural language processing and opens new directions for quantum AI research.

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