

RaDeR: Reasoning-aware Dense Retrieval Models

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Abstract

We propose RaDeR, a set of reasoning-based dense retrieval models trained with data derived from mathematical problem solving using large language models (LLMs). Our method leverages retrieval-augmented reasoning trajectories of an LLM and self-reflective relevance evaluation, enabling the creation of both diverse and hard-negative samples for reasoning-intensive relevance. RaDeR retrievers, trained for mathematical reasoning, effectively generalize to diverse reasoning tasks in the BRIGHT and RAR-b benchmarks, consistently outperforming strong baselines in overall performance. Notably, RaDeR achieves significantly higher performance than baselines on the *Math* and *Coding* splits. In addition, RaDeR presents the first dense retriever that outperforms BM25 when queries are Chain-of-Thought reasoning steps, underscoring the critical role of reasoning-based retrieval to augment reasoning language models. Furthermore, RaDeR achieves comparable or superior performance while using only 2.5% of the training data used by the concurrent work REASONIR, highlighting the quality of our synthesized training data. Our code, data, and retrieval models are publicly available.¹

1 Introduction

Large language models (LLMs) have demonstrated impressive reasoning capabilities on a wide range of tasks. Yet, they often benefit from retrieval augmentation to enhance accuracy, attributability (Asai et al., 2024), and the interpretability (Nakano et al., 2022) of their outputs. Retrieval models for LLM augmentation generally perform reasonably well at lexical and semantic term matching, however they face challenges when reasoning is needed for relevance prediction (Su et al., 2024).

¹Code: <https://github.com/Debrup-61/RaDeR>
Models: <https://huggingface.co/Raderspace>

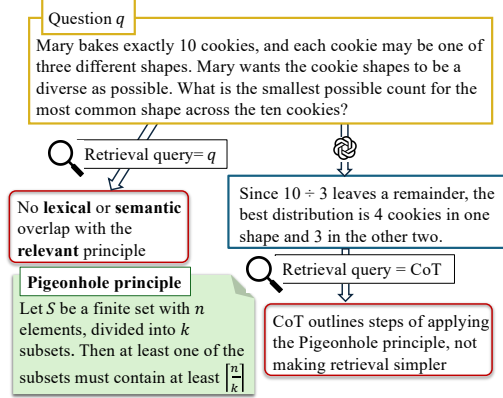


Figure 1: An example based on sample ‘TheoremQA_jianyuxu/pigeonhole3’ of BRIGHT, where term matching retrievers face challenges in retrieving the relevant theorem w.r.t. both questions and CoT reasoning.

Recent works have tried to address the reasoning limitation of existing models for relevance prediction. Two main approaches have emerged: (1) interleaved reasoning and retrieval (Hu et al., 2025; Jin et al., 2025; Song et al., 2025), and (2) reasoning-based re-ranking models (Weller et al., 2025; Samarinas and Zamani, 2025). While the first group is more effective than standard RAG (Lewis et al., 2021; Gao et al., 2024; Guu et al., 2020; Shi et al., 2024), they are limited to the reasoning steps of LLMs for retrieval. As the example in Figure 1 shows, the reasoning steps of LLMs may not align with those needed for retrieval. Solving the question in this example requires retrieving the *pigeonhole principle*, where there are no matching terms between the question and the principle. The reasoning steps by GPT-4 also do not simplify the retrieval of the pigeonhole principle, since they outline the steps of applying the pigeonhole principle to solve the question. On the other hand, reranking models are inherently limited by the candidate set produced by the first-stage retriever. To the best of our knowledge, there are no *first-stage reasoning-based retrieval* models.

Developing reasoning-based retrievers poses multiple *challenges*. The primary challenge is automatic generation of diverse and high-quality training data. Specifically, training data should include queries of *diverse formats and lengths* as well as samples with varying *degrees of reasoning complexity*. Beyond this, training retrieval models using representation learning (Dai et al., 2023; Karpukhin et al., 2020; Izacard et al., 2022) presents the additional challenge of generating hard-negative reasoning samples.

We propose RaDeR, a set of first-stage reasoning-based retrieval models trained with synthesized data from mathematical reasoning. Specifically, we use an LLM for mathematical problem solving with a retrieval-augmented search-based reasoning approach, where the LLM can retrieve and apply theorems needed for solving intermediate subproblems. To generate training data, we then sample reasoning trajectories with retrieval nodes based on the assumption that information retrieved during intermediate steps of the LLM’s search process is likely to be relevant to the original question. This approach, illustrated in Figure 2, naturally yields a diverse set of queries, varying in length and complexity.

In addition, verifying LLM-generated answers to mathematical questions provides a proxy for evaluating the relevance of retrieved information. To further ensure the quality of generated training data, the relevance evaluation is enhanced by *self-reflection*. These evaluations help with generation of high-quality data. Additionally, any retrieved theorem evaluated as non-relevant is considered as a hard-negative sample for reasoning.

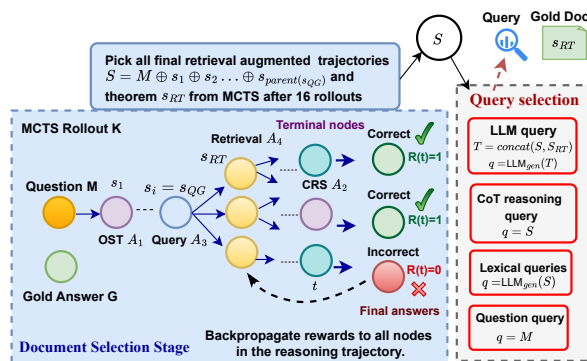


Figure 2: An overview of the **RaDeR** data generation pipeline. The *OST* action stands for *one step thought* generation, and *CRS* stands for *complete remaining solution* steps action.

We perform a comprehensive evaluation of

RaDeR models, including evaluation of retrieval performance on reasoning-intensive benchmarks, traditional benchmarks mainly requiring term matching, as well as evaluation of QA performance using retrieval augmentation. Experimental results on BRIGHT (Su et al., 2024) show that RaDeR outperforms strong baselines by at least **2** points in nDCG across various settings. These findings demonstrate that training retrieval models for mathematical reasoning effectively generalizes to other types of reasoning for information retrieval. In addition to improvements in overall performance, RaDeR demonstrates particularly strong performance in *Math* and *Coding* splits of BRIGHT. We observe nDCG@10 relative improvements of **37-40%** in the theorem-Q split, and **8-26%** over the Leet coding split. RaDeR presents the *first* dense retriever that outperforms BM25 in a zero-shot setting where reasoning steps are used as retrieval queries. This achievement provides strong evidence for the necessity of reasoning-based retrievers even when retrievers augment reasoning language models. On the MMTEB (Enevoldsen et al., 2025) reasoning subset, RAR-b (Xiao et al., 2024), RaDeR significantly outperforms all sparse and open-source models, performing on par with large proprietary models such as OpenAI-3-large. In a concurrent work, Shao et al. (2025) also trains reasoning-based first-stage retrieval models REASONIR. RaDeR achieves a **1.1** point increase in nDCG@10 performance compared to REASONIR, corresponding to a **4.5%** relative gain. Performance of RaDeR is particularly significant given that it is trained with 43,120 samples, about **2.5%** of samples used for REASONIR (1,729,368), demonstrating the *effectiveness* of our synthesized data.

In summary our main contributions are: (1) We introduce RaDeR, one of the first set of reasoning-based first-stage retrieval models, which achieves state-of-the-art performance on the BRIGHT benchmark, (2) We propose a training data synthesis strategy for *reasoning-aware relevance ranking* using mathematical QA datasets consisting of only questions and their gold answers, *without requiring any notion of document relevance*, (3) Our models, though trained exclusively with data derived from the training sets of mathematical benchmarks, demonstrate *strong generalizability* to diverse splits of BRIGHT unrelated to math such as leetcode, stackoverflow, psychology, as well as to TREC DL’ (Craswell et al., 2020) and ’20 (Craswell et al., 2021) datasets which are not reasoning intensive.

2 Related Works

Interleaving reasoning with retrieval. Previous works have explored interleaving chain-of-thought (CoT) reasoning with retrieval using *off-the-shelf* retrievers (Trivedi et al., 2023; Shao et al., 2023; Yao et al., 2023b; Schick et al., 2023) (see Appendix A.2 for details). In this paradigm, a *retrieval action* at a given step leverages a subset of the previously generated CoT reasoning steps as the *query* for retrieval. However, these methods are limited by their reliance on off-the-shelf retrievers.

Recent works like Search-R1 (Jin et al., 2025) and R1-Searcher (Song et al., 2025) use reinforcement learning to optimize **reasoning-based query generation** while keeping the retrieval model fixed. In complex retrieval tasks, direct lexical or semantic overlap between the initial question and the relevant document is often limited. By incorporating intermediate CoT reasoning during retrieval, the system can better bridge this gap.

Recent research explores **search-based methods** for exploring the space of CoT reasoning paths, such as random sampling (Wang et al., 2023b) and Monte Carlo Tree Search (MCTS) (Qi et al., 2025; Yao et al., 2023a; Guan et al., 2025; Hao et al., 2023). Retrieval-augmented generation (RAG) has also been integrated into MCTS-based frameworks by introducing retrieval-related *actions* such as query generation and document retrieval (Tran et al., 2024; Hu et al., 2025).

Reasoning-based re-rankers. Recent methods such as RANK-1 (Weller et al., 2025) and InterRank (Samarinas and Zamani, 2025) train re-ranking models using knowledge distillation from reasoning LLMs and reinforcement learning respectively (see Appendix A.3 for details). However, these approaches remain limited by the candidate set of initial retrievers.

Data augmentation for IR. Existing methods (Nogueira et al., 2020; Bonifacio et al., 2022; Dai et al., 2023) expand queries or documents with likely terms and generate new queries from existing documents. Recent methods (Hu et al., 2024; Lee et al., 2024) leverage LLMs to build iterative pipelines for synthetic data generation - using LLM-as-judge as the primary signal for data quality. The concurrent work ReasonIR (Shao et al., 2025) also targets reasoning-intensive search, but generates queries without labeled scalar rewards.

Our work is the first to generate synthetic datasets specifically tailored for *reasoning-*

intensive information retrieval. Unlike previous methods, RaDeR directly integrates reasoning into first-stage retrieval, leveraging MCTS to create sample-efficient synthetic data.

3 RaDeR: Reasoning-aware Retrievers

We propose a framework that includes a first-stage retriever and a re-ranking model, both performing reasoning to predict relevance. For the **first-stage retriever**, we adopt a uni-embedding bi-encoder architecture of dense retrieval models (Lei et al., 2023). For **re-ranking**, we fine-tune a pointwise *cross-attention* model that takes the concatenation of the query and document. Our re-ranker directly predicts relevance scores, while existing reasoning-based re-rankers (Weller et al., 2025; Samarinas and Zamani, 2025) rely on test-time compute for reasoning, making our approach significantly more efficient at inference time.

4 Generating Retrieval Training Data

The main challenge of developing reasoning-based retrieval models is synthesizing effective data that includes queries of *diverse formats and lengths*, requiring varying degrees of reasoning complexity. This diversity is essential for adaptive RAG systems (Asai et al., 2023; Hu et al., 2025; Jin et al., 2025), where LLMs may call retrievers at any intermediate reasoning or solution step.

A widely used data augmentation technique in IR involves prompting LLMs with passages to generate relevant queries (Dai et al., 2023; Bonifacio et al., 2022; Lee et al., 2024; Hu et al., 2024). However, this technique can limit the diversity of generated queries. It also poses challenges in verifying that the generated queries are relevant to the given passages through reasoning and are not generic, especially in the absence of ground truth for either the relevance reasoning steps or the queries.

To address the aforementioned challenges, we propose to generate training data using a retrieval-augmented Monte Carlo Tree Search (MCTS) reasoning approach (Kocsis and Szepesvári, 2006) to solve mathematical problems by LLMs. Our motivation is twofold. First, solving mathematical problems often requires applying theorems to subproblems, which enables the integration of retrievers. Theorems found to be relevant to subproblems are also relevant to the original question due to the reasoning steps that connect them. Second, verifying LLM answers against gold answers provides

a proxy for evaluating the utility of retrieved theorems in solving subproblems.

4.1 Retrieval-Augmented Search-based Reasoning

We use a framework for solving mathematical problems by LLMs using a Monte Carlo Tree Search (MCTS) process, augmented with retrieval over a collection of theorems and guided by scalar feedback based on the gold answers.

MCTS overview. To solve a math problem M from a given dataset, the MCTS algorithm prompts an LLM denoted by LLM_{gen} to incrementally build a search tree that explores possible reasoning trajectories toward the final answer. The generation process in MCTS is driven by two basic components: an *action space* A and a *reward function* R . The root node of the tree corresponds to the input question M . An edge from each node represents an action $a_i \in A$, and the resulting child node is an intermediate reasoning step s_i , which is generated by applying action a_i to the current reasoning trajectory $M \oplus s_1 \oplus s_2 \oplus \dots \oplus s_{i-1}$. Each node in the tree is assigned a value $Q(s, a)$, which represents the expected *reward* on taking action a from node s . Initially, all nodes have $Q(s, a) = 0$, resulting in a random tree exploration. As the algorithm performs rollouts, the Q values of the nodes are updated based on the rewards R and the search is guided toward better reasoning trajectories.

Action space. We extend the rStar framework (Qi et al., 2025) by adding two new actions for query generation and retrieval. As part of the *retrieval action*, RaDeR performs two additional steps: *self-reflection* which evaluates the relevance of retrieved theorems to the current problem context, and *self-summarization* which generates concise natural-language summaries of the theorems to facilitate their integration into subsequent reasoning steps. The action space A is described in detail below:

A_1 : Propose One-Step Thought (OST). This action uses LLM_{gen} to generate a single CoT reasoning step s_i based on the context $M \oplus s_1 \oplus s_2 \oplus \dots \oplus s_{i-1}$. The result node can be a *terminal* node if s_i contains the answer in *boxed* notation (prompts in Appendix M).

A_2 : Propose Complete Reasoning Steps (CRS). This action uses LLM_{gen} to complete the full solution by generating s_i containing multiple reasoning steps, based on the current context $M \oplus s_1 \oplus s_2 \oplus \dots \oplus s_{i-1}$. The result node is always a *terminal*

node of the MCTS. Prompt details are provided in Appendix N.

A_3 : Generate Query (QG). This action prompts LLM_{gen} to generate a retrieval query node s_{QG} based on the current context $M \oplus s_1 \oplus s_2 \oplus \dots \oplus s_{\text{parent}(\text{QG})}$. A *generate query* action is always followed by a *retrieve theorem* action. For action A_3 , we use few-shot prompting, details in Appendix O.

A_4 : Retrieve Theorem (RT). This action uses a retriever to obtain the top- k theorems s_{RT} from the theorem corpus using the query s_{QG} of its parent node. Each of the k retrieved theorems that pass self-reflection is then added as a child node s_{RT} .

Self-Reflection and summarization. To avoid expanding the search tree with irrelevant theorem nodes, we adopt *self-reflection* (Asai et al., 2023; Xia et al., 2025) to evaluate the relevance of retrieved theorems. We use few-shot prompting to guide LLM_{gen} in generating both a relevance label (“relevant” or “non-relevant”) and a supporting explanation. The input comprises the original math question M , the current intermediate solution path $(M \oplus s_1 \oplus s_2 \oplus \dots \oplus s_{\text{parent}(\text{QG})})$, and the retrieved theorem s_{RT} . Only theorems labeled as relevant are added as nodes in the MCTS tree, while non-relevant ones are pruned early to reduce unnecessary expansion and computation. This mechanism enables RaDeR to focus exploration on more promising retrieval-augmented solution trajectories. We provide the prompts used for *self-reflection* and *self-summarization* in Appendix P.

Sampling solutions with MCTS rollouts. We sample multiple candidate solution trajectories with MCTS rollouts following rStar (Qi et al., 2025). The details are described in Appendix B.

The **reward function** $R(t)$ for a terminal node t is calculated based on whether the solution trajectory reaches the correct answer to the input question M . Specifically, if the trajectory leads to the correct answer, $R(t) = 1$; otherwise, $R(t) = 0$.

4.2 Synthesizing Training Data

To build training data for reasoning-based retrievers, we sample high-reward solution trajectories generated by the MCTS framework. For a math question M , we extract all solution trajectories that contain at least one retrieval node, denoted as $S = M \oplus s_1 \oplus s_2 \oplus \dots \oplus s_{\text{QG}} \oplus s_{\text{RT}} \oplus \dots \oplus s_t$. From each selected solution trajectory S , we generate training samples in the form (q, p, N) , where q denotes a query, p is a positive theorem relevant to q , and N is a set of hard-negative theorems for

reasoning-intensive retrieval. The retrieved theorem s_{RT} in S is used as the positive document p of the generated samples. Any theorem retrieved with respect to s_{QG} that was not labeled as relevant in self-reflection is included in the set of hard negatives N .

Reasoning-intensive data. To generate reasoning samples, the set of positive and hard-negative theorems is paired with three types of queries. (1) *MCTS CoT reasoning queries*: we use the CoT reasoning steps (partial solution) up to the query node, $M \oplus s_1 \oplus s_2 \oplus \dots \oplus s_{\text{parent}(QG)}$, as the retrieval query. Note that we do not use the query s_{QG} generated by MCTS for retrieval training, since the specific prompt format for their generation may limit the diversity of our training data. These queries are denoted as q_{CoT} . (2) *LLM reasoning queries*: We prompt the LLM_{gen} to generate a query based on the math question M , the reasoning steps up to the query node (excluding the query), and the theorem s_{RT} . This group of queries is referred to as q_{llmq} . Using few-shot prompting, the LLM_{gen} is guided to generate reasoning-intensive queries that have low lexical and semantic term overlap with the positive theorems. Unlike the query generated during the QG action of MCTS, which directs LLM_{gen} to produce a hypothetical theorem for the current subproblem, q_{llmq} is a concise query that directly reflects the information need of the subproblem. The prompt template for this query is provided in Appendix Q. (3) *Questions as queries*: the input math question M is used as the query for retrieving theorem s_{RT} . Given the low lexical and semantic overlap between M and s_{RT} , it serves as a good reasoning sample for training. We denote this type of queries as q_{question} .

Term-matching data. Training a reasoning-based retriever should not hurt the performance of queries that can be addressed with lexical/semantic term matching. To achieve this, we follow the widely used approach of synthetically generating training samples for retrieval models. Specifically, to generate queries that have high term similarity with their respective relevant theorems, we prompt LLM_{gen} using only the theorem s_{RT} . Following Promptagator (Dai et al., 2023), we add a filtering step based on *round-trip consistency*. Only the queries for which BM25 retrieves the corresponding positive theorem s_{RT} within its top- k ($k=20$) results are kept. This filtering step typically results in queries with large term overlap. Hard negatives are extracted from top results of BM25 and Re-

pLLaMA retrieval models. We denote this type of queries as q_{lexical} .

Synthesized data mix. We construct our retrieval training dataset by combining all query types, including both reasoning-based and lexical queries. Detailed statistics of the synthesized training data are provided in Appendix E.

5 Experimental Settings

Base language models. To train dense retrieval models, we primarily utilize the instruction-tuned variants of the Qwen2.5 suite of LLMs (Yang et al., 2024). For ablation studies on model size, we train a series of Qwen-2.5-instruct models with varying parameter sizes ranging from 3B, 7B to 14B. We additionally perform ablation studies using different LLMs including gte-Qwen2-7B-Instruct (Li et al., 2023a) and Llama-3.1-Instruct (Grattafiori et al., 2024).

Math datasets for data generation. We incorporate two datasets for mathematical problem solving in RaDeR: MATH (Hendrycks et al., 2021) and a subset of examples from NuminaMath (Li et al., 2024). Details are provided in Appendix C.

Retrieval-augmented MCTS. We apply the RaDeR framework using a fixed set of parameters, as detailed in Appendix D. For each input math question in a dataset for mathematical problem solving, we perform 16 rollouts. We use Re-pLLaMA (Ma et al., 2023) as the retriever in our MCTS algorithm. The retrieval corpus consists of formal mathematical theorems from ProofWiki,² which is also used in the BRIGHT benchmark (Su et al., 2024).

Training details. We train both the *retriever* and the *re-ranker* for reasoning-intensive relevance prediction. The retriever is trained using the standard contrastive *InfoNCE* loss using in-batch negatives in addition to the hard negatives from synthesized training samples. We use 12 hard negatives per query and treat passages from other examples in the batch as in-batch negatives. The retriever training details are presented in Appendix F. Similar to the retriever, the reranker is trained using contrastive loss. The reranker training details are presented in Appendix G. The hyperparameters used for training are presented in Table 12.

²<https://proofwiki.org/> — a comprehensive collection of over 20K formal definitions and theorem proofs

	StackExchange							Coding		Theorem-based			Avg.
	Bio	Earth	Econ	Psy	Rob	Stack	Sus	Leet	Pony	AoPS	TheoQ	TheoT	
Sparse and Open-source Baselines													
BM25*	18.9	27.2	14.9	12.5	13.6	18.4	15.0	24.4	7.9	6.2	10.4	4.9	14.5
Qwen*	30.6	36.4	17.8	24.6	13.2	<u>22.2</u>	14.8	25.5	9.9	<u>14.4</u>	27.8	32.9	22.5
Qwen2	<u>34.1</u>	42.6	18.2	27.4	13.2	17.3	<u>20.9</u>	30.4	2.2	13.3	30.6	32.6	23.5
GritLM*	24.8	32.3	18.9	19.8	<u>17.1</u>	13.6	17.8	29.9	22.0	8.8	25.2	21.2	21.0
Inst-XL*	21.6	34.3	22.4	27.4	18.2	21.2	19.1	27.5	5.0	8.5	15.6	5.9	18.9
E5*	18.6	26.0	15.5	15.8	16.3	11.2	18.1	28.7	4.9	7.1	26.1	26.8	17.9
Proprietary Baselines													
Google*	22.7	34.8	19.6	27.8	15.7	20.1	17.1	29.6	3.6	9.3	23.8	15.9	20.0
Voyage*	23.1	25.4	19.9	24.9	10.8	16.3	15.4	30.6	1.5	7.5	27.4	11.6	17.9
RaDeR Models (MATH dataset, ($q_{lmq} + q_{CoT} + q_{lexical}$))													
Qwen2.5-7B-instruct	25.4	30.0	16.7	25.3	14.0	21.3	16.3	37.0	8.2	15.7	42.7	44.4	<u>24.6</u>
gte-Qwen2-7B	34.6	38.9	<u>22.1</u>	33.0	14.8	22.5	23.7	37.3	5.0	10.2	28.4	35.1	25.5
Llama3.1-8B-Instruct	29.3	27.3	<u>17.5</u>	<u>28.2</u>	12.1	18.2	16.1	38.6	<u>11.8</u>	6.4	32.3	33.1	22.6
RaDeR Models (MATH+NuminaMath datasets, all query types)													
Qwen2.5-7B-instruct	25.1	28.3	18.2	25.9	15.3	22.2	16.3	35.9	6.9	10.4	<u>40.8</u>	47.1	24.4

Table 1: nDCG@10 performance of strong baselines and our models over the BRIGHT benchmark using the original question as queries for retrieval. Results of models with * are taken from (Su et al., 2024).

	StackExchange							Coding		Theorem-based			Avg.
	Bio	Earth	Econ	Psy	Rob	Stack	Sus	Leet	Pony	AoPS	TheoQ	TheoT	
Sparse and Open-source Baselines													
BM25*	53.6	54.1	24.3	<u>38.7</u>	<u>18.9</u>	27.7	<u>26.3</u>	19.3	17.6	3.9	19.2	20.8	27.0
Qwen*	35.5	43.1	24.3	33.4	15.4	22.9	23.9	25.4	5.2	4.6	28.7	34.6	24.8
Qwen2	38.3	47.3	24.0	35.2	15.9	23.3	27.9	29.5	8.9	2.9	30.8	35.1	26.6
GritLM*	33.3	39.1	22.4	28.9	17.4	21.3	24.1	31.9	12.0	6.7	27.3	30.1	24.5
Inst-XL*	<u>46.7</u>	<u>51.2</u>	29.9	40.5	20.8	30.1	26.9	35.1	2.1	8.2	24.2	17.0	26.9
E5*	29.3	43.9	19.9	26.6	11.6	19.8	15.6	29.1	0.9	5.3	27.0	36.6	22.1
Proprietary Baselines													
Google*	36.4	45.6	<u>25.6</u>	38.2	18.7	<u>29.5</u>	15.7	31.1	3.7	10.0	27.8	30.4	26.2
Voyage AI*	36.7	42.8	24.6	34.2	13.7	24.2	21.7	31.4	2.2	6.6	30.3	28.1	24.7
RaDeR Models (MATH dataset, $q_{llmq} + q_{CoT} + q_{lexical}$)													
Qwen2.5-7B-instruct	32.4	38.0	21.5	33.2	14.5	25.5	18.1	30.1	14.0	<u>11.4</u>	42.1	47.2	27.3
gte-Qwen2-7B	36.1	42.9	25.2	37.9	16.6	27.4	25.0	34.8	11.9	12.0	37.7	43.4	29.2
Llama3.1-8B-Instruct	37.6	41.4	21.1	33.1	12.5	27.7	15.8	<u>35.0</u>	23.6	7.1	36.9	40.5	<u>27.7</u>
RaDeR Models (MATH+NuminaMath datasets, all query types)													
Qwen2.5-7B-instruct	37.5	40.0	19.3	31.1	14.1	25.8	17.6	27.2	<u>18.7</u>	9.9	40.4	<u>43.8</u>	27.4

Table 2: nDCG@10 performance using GPT-4 CoT reasoning as queries for retrieval over BRIGHT. Results of models with * are taken from (Su et al., 2024).

6 Experimental Results

We comprehensively evaluate our RaDeR retriever and re-ranker models on (1) the reasoning-intensive benchmark BRIGHT (Su et al., 2024) and MMTEB reasoning tasks based on RAR-b (Xiao et al., 2024), and (2) widely used benchmark MS MARCO (Bajaj et al., 2018) to measure term matching performance. We also evaluate the performance of reasoning LLMs when augmented with our RaDeR models. Following previous studies (Su et al., 2024), nDCG at top-10 (nDCG@10) is used as the evaluation metric to compare different models. We also report the recall and precision of retrieval models in Appendix I.

6.1 BRIGHT Retrieval Performance

We first present the performance of our RaDeR retrieval models on the BRIGHT benchmark (Su et al., 2024). Following the benchmark, we compare our models against a diverse set of baselines including (1) BM25 (Robertson et al., 1995),

(2) open-source dense retrieval models Instructor-XL (Su et al., 2023), E5-Mistral (Wang et al., 2024), GritLM (Muennighoff et al., 2024), gte-Qwen1.5 (Li et al., 2023a), gte-Qwen2 (Li et al., 2023a), and (3) proprietary models from Voyage (Voyage) and Google (Lee et al., 2024).

Following the benchmark, all models are evaluated in two settings: (1) using the original questions from BRIGHT as retrieval queries, and (2) using CoT reasoning steps generated by GPT-4, included in the benchmark, as retrieval queries. Performance results for the two settings are reported in Tables 1 and 2, respectively.

Overall performance. RaDeR-gte-Qwen2-7B achieves the best average performance of **25.5** on BRIGHT, outperforming strong baselines by at least **2** points in both query settings. These results demonstrate that training retrieval models for mathematical reasoning *generalizes* to other types of reasoning required for different retrieval tasks.

Theorem-based splits. Compared to baselines,

Data of RaDeR-	StackExchange							Coding		Theorem-based			Avg.
	Bio	Earth	Econ	Psy	Rob	Stack	Sus	Leet	Pony	AoPS	TheoQ	TheoT	
q_{lexical}	23.2	32.2	16.5	23.4	13.9	16.9	17.2	31.6	7.6	8.3	31.8	24.2	20.5
q_{llmq}	25.8	32.0	15.8	23.3	14.8	18.7	15.7	29.9	16.4	13.0	38.8	35.4	23.3
$q_{\text{llmq}} + q_{\text{CoT}}$	24.5	26.9	17.0	25.6	14.8	21.1	17.1	30.6	9.9	12.8	42.5	38.5	23.4
$q_{\text{llmq}} + q_{\text{CoT}} + q_{\text{lexical}}$	25.4	30.0	16.7	25.3	14.0	21.3	16.3	37.0	8.2	15.7	42.7	44.4	24.6

Table 3: nDCG@10 performance of RaDeR on BRIGHT when retrievers are trained using different types of samples generated from MATH. The original math question is used as the query. RaDeR Qwen2.5-7B-instruct models are used for the ablations.

RaDeR Models (all query types)	StackExchange							Coding		Theorem-based			Avg.
	Bio	Earth	Econ	Psy	Rob	Stack	Sus	Leet	Pony	AoPS	TheoQ	TheoT	
Qwen2.5-3B-instruct	26.9	29.7	16.5	24.8	13.3	19.3	15.2	37.5	10.6	9.4	34.9	36.9	22.9
Qwen2.5-7B-instruct	25.1	28.3	18.2	25.9	15.3	22.2	16.3	35.9	6.9	10.4	40.8	47.1	24.4
Qwen2.5-14B-instruct	30.9	31.5	17.3	27.9	16.0	24.2	16.4	40.9	10.2	12.7	42.6	45.8	26.4

Table 4: nDCG@10 performance of RaDeR on BRIGHT when retrievers are trained using different sizes of Qwen-2.5-instruct models (3B, 7B, and 14B). The original math question is used as the query.

RaDeR achieves the largest improvements on the TheoT and TheoQ splits in both settings, as shown in Tables 1 and 2. For example, RaDeR models improve the performance of TheoQ by **12.1** and **11.3** points when questions and CoT reasoning are used as queries, respectively. Although these splits of BRIGHT primarily require mathematical reasoning, the performance improvements on TheoQ are particularly noteworthy since the retrieval collection for this split consists of similar questions, a format not seen during the training of RaDeR models. The strong performance of RaDeR on this split demonstrates its potential for tasks such as demonstration selection in in-context learning, a task shown to have significant impacts on the performance of LLMs (Rubin et al., 2022).

Coding splits. Results in Table 1 show that RaDeR achieves the best performance on *LeetCode*, surpassing the strong baselines by **8** points. Results on *Pony* in the CoT reasoning setting, Table 2, also show improvements of RaDeR over the strongest baseline. These results are particularly significant, as they demonstrate that training our retrievers for mathematical reasoning yields substantial improvements in code retrieval, despite not having any code-specific training data. Our qualitative analysis reveals that in most cases, coding problems rely on solving underlying mathematical subproblems. We believe that the ability to recognize this mathematical substructure enables RaDeR models to enhance retrieval effectiveness on these coding splits.

Ablation studies. We investigated the performance of RaDeR retrievers under two ablation settings: (1) training with different subsets of query types in the synthesized data, and (2) using base

LLM of different sizes as the retriever encoder. Results in Table 3 highlight the *complementary* role of the diverse query types in our synthesized training data. Additionally, Table 4 shows consistent gains from model scaling, with the Qwen2.5-14B-instruct model outperforming both the 7B variant and the 4B variant by 2 and 3.5 nDCG points, respectively.

6.2 Retrieval Performance on RAR-b

We evaluate the performance of RaDeR on the reasoning-retrieval tasks of MMTEB (Enevoldsen et al., 2025) which are based on the Math and Coding splits of the RAR-b (Xiao et al., 2024). As

Model	Math	Coding
Open-source Baselines		
Contriever (w/ Inst.)	0.218	0.071
all-mpnet-base-v2 (w/ Inst.)	0.692	0.488
all-MiniLM-L6-v2 (w/ Inst.)	0.624	0.423
Dragon+ (w/ Inst.)	0.362	0.128
Instructor-XL (w/ Inst.)	0.580	0.495
bge-large (w/ Inst.)	0.498	0.453
E5-Mistral (w/ Inst.)	0.740	0.785
GritLM (w/ Inst.)	0.824	<u>0.838</u>
Proprietary Models		
Cohere-Embed-v3 (w/ Inst.)	0.721	0.566
OpenAI-ada-002 (w/ Inst.)	0.673	0.824
OpenAI-3-large (w/ Inst.)	0.877	0.894
RaDeR model (MATH, all query types)		
gte-Qwen2-7B-instruct (w/Inst.)	<u>0.852</u>	0.835

Table 5: nDCG@10 performance of retrievers on the Math and Coding splits of RAR-b. Performance of baselines are from (Xiao et al., 2024).

shown in Table 5, our model achieves performance comparable to the strongest open-source baseline, outperforming on the MATH split. Furthermore, it demonstrates performance that is comparable to or surpasses that of top-performing closed-source models. These results once again highlight the

strong generalizability of RaDeR across diverse reasoning-based retrieval tasks.

Model	DEV		DL19	DL20
	MRR@10	R@1k	nDCG@10	nDCG@10
BM25	18.4	85.3	50.6	48.0
ANCE	33.0	95.9	64.5	64.6
CoCondenser	38.2	98.4	71.7	68.4
TAS-B	34.0	97.5	71.2	69.3
GTR-base	36.6	98.3	-	-
GTR-XXL	38.8	99.0	-	-
OpenAI Ada2	34.4	98.6	70.4	67.6
bi-SimLM	39.1	98.6	69.8	69.2
RepLLaMA	41.2	99.4	74.3	72.1
SimLM	41.1	98.7	71.4	69.7
RaDeR	34.4	98.1	71.2	<u>70.7</u>

Table 6: Performance of baselines and RaDeR-gte-Qwen2-7B (trained with MATH) on MS MARCO. Performance of baselines are from (Ma et al., 2023; Wang et al., 2023a).

6.3 Retrieval on Traditional IR Benchmarks

Table 6 presents the performance of RaDeR and strong baselines on the MS MARCO passage retrieval task (Bajaj et al., 2018). We evaluate on the official small subset of the MS MARCO development set, following RepLLaMA (Ma et al., 2023), as well as TREC-DL’19 (Craswell et al., 2020) and TREC-DL’20 (Craswell et al., 2021). These test sets primarily rely on term matching and do not require complex reasoning. Our model based on gte-Qwen2-7B trained with llmq, CoT, and lexical queries from MATH, demonstrates competitive performance with strong baselines. These results indicate that training our retrievers for reasoning-intensive tasks does not compromise their effectiveness on standard IR benchmarks where reasoning is not necessary.

6.4 BRIGHT Reranking Performance

We report the performance of our RaDeR rerankers on the reasoning-intensive BRIGHT benchmark. Table 8 shows the performance of reranking top-10 and top-100 of BM25 results using questions as queries. Our RaDeR rerankers based on Qwen2.5 and gte-Qwen2, both trained on all query types from MATH, outperform GPT-4 in the top-100 reranking setting by **+2.5** and **+2.1** nDCG points, respectively. The most significant improvements are observed on the LeetCode and TheoremQA-questions splits, with gains of **+20.3** and **+20.4** nDCG points, respectively.

Table 9 presents the performance of reranking BM25 results using GPT-4o CoT reasoning, where

the rerankers receive only the question as the query. We compare RaDeR rerankers against strong baselines including RankLLaMA (Ma et al., 2023), Mono-T5-3B, and RANK1 (Weller et al., 2025). RANK1, the strongest baseline, is trained on 635K examples from MS MARCO, and utilizes test-time compute to perform reasoning before relevance prediction. In contrast, RaDeR models are trained with substantially fewer samples on mathematical reasoning, 43K from MATH and 78K from NuminaMATH, yet they achieve highly competitive performance. In addition, RaDeR models are significantly more computationally efficient since they only generate relevance scores.

Method	Accuracy(%)
Base model (no retrieval)	71.0
In-context RAG with RaDeR	75.0
In-context RAG with RepLLaMA	72.6
In-context RAG with gold theorems	77.6
MCTS with only OST action	75.0
MCTS with OST + RepLLaMA retrieval	78.9
MCTS with OST + RaDeR retrieval	80.2
MCTS with OST + gold theorems	81.5

Table 7: QA performance of Qwen-2.5-7B-Instruct on the TheoremQA theorems split of BRIGHT in different settings of retrieval augmentation.

6.5 RAG Performance using RaDeR

Table 7 provides the answer accuracy of Qwen-2.5-7B-Instruct on the TheoremQA split of BRIGHT in two settings of retrieval augmentation. First setting is in-context RAG where the input question is used as the query to retrieval models. We also perform evaluation using the retrieval-augmented MCTS framework, employing majority voting as the strategy for answer selection. In both settings, augmenting with the results of RaDeR outperforms augmentation with RepLLaMA. These results demonstrate the impact of augmenting strong LLMs with reasoning-based retrievers.

7 Analysis

We present examples from our synthesized training data, along with qualitative examples of RaDeR from the BRIGHT evaluation. We highlight representative cases where RaDeR successfully retrieves relevant information, as well as illustrative failure cases to analyze its limitations.

Examples from Synthesized Training Data.

We present an example from our training data, illustrating different query types, q_{llmq} , q_{CoT} , and

Reranker	top-k	StackExchange							Coding		Theorem-based			Avg.
		Bio.	Earth.	Econ.	Psy.	Rob.	Stack.	Sus.	Leet.	Pony	AoPS	TheoQ.	TheoT.	
None*	-	19.2	27.1	14.9	12.5	13.5	16.5	15.2	<u>24.4</u>	7.9	6.2	9.8	4.8	14.3
MiniLM*	100	8.5	18.9	6.0	5.4	7.6	7.9	8.9	15.0	11.3	6.1	3.6	0.5	8.3
Gemini*	10	21.9	29.7	16.9	14.2	16.1	16.7	16.7	24.5	8.0	6.2	9.5	8.2	15.7
GPT-4*	100	33.8	34.2	16.7	<u>27.0</u>	22.3	27.7	11.1	3.4	15.6	1.2	2.0	8.6	17.0
RaDeR models (MATH, all query types)														
Qwen2.5-7B-instruct	100	<u>26.9</u>	30.6	17.0	24.9	18.2	17.8	<u>20.5</u>	23.7	14.3	<u>4.4</u>	22.4	<u>13.6</u>	<u>19.5</u>
gte-Qwen2-7B-instruct	100	26.1	30.4	16.8	26.6	18.7	18.5	16.5	18.7	20.8	2.9	20.4	12.4	19.1
RaDeR models (MATH+NuminaMath, all query types)														
gte-Qwen2-7B-instruct	100	25.5	<u>31.8</u>	19.3	28.8	<u>22.0</u>	<u>19.8</u>	20.1	17.1	11.9	1.6	18.9	14.3	19.3
Qwen2.5-7B-instruct	100	25.0	31.1	<u>17.3</u>	26.4	21.1	19.5	21.3	21.5	<u>16.1</u>	5.1	<u>21.7</u>	14.3	20.0

Table 8: nDCG@10 performance of different rerankers on BRIGHT. Reranking is performed on the top-10 or top-100 results retrieved using BM25 with the question as the query. Results with * are from (Su et al., 2024).

	StackExchange							Coding		Theorem-based			Avg.
	Bio.	Earth.	Econ.	Psy.	Rob.	Stack.	Sus.	Leet.	Pony	AoPS	TheoQ.	TheoT.	
BM25*	19.2	27.1	14.9	12.5	13.5	16.5	15.2	24.4	7.9	6.0	13.0	6.9	14.8
BM25 on GPT-4o CoT	53.6	53.6	24.3	38.6	18.8	22.7	25.9	<u>19.3</u>	17.7	3.9	18.9	20.2	26.5
Reranking on GPT-4o CoT k=100													
MonoT5-3B*	16.0	24.0	17.7	19.5	8.0	10.5	19.5	17.2	29.2	7.1	20.3	12.0	16.8
RankLLaMA-7B*	17.5	15.5	13.1	13.6	17.9	6.9	16.9	8.4	46.8	2.2	4.5	3.5	13.9
Rank1-7B*	<u>48.8</u>	<u>36.7</u>	20.8	35.0	22.0	18.7	36.2	12.7	31.2	<u>6.3</u>	23.7	<u>37.8</u>	27.5
RaDeR Models Reranking on GPT-4o CoT results k=100 (MATH, all query types)													
Qwen2.5-7B-instruct	40.8	31.8	<u>25.0</u>	39.7	21.6	25.7	27.2	17.3	29.9	1.6	22.4	36.9	<u>26.7</u>
gte-Qwen2-7B-instruct	36.0	29.3	23.2	40.0	23.2	24.1	22.2	17.8	<u>34.9</u>	1.5	20.4	35.3	25.7
RaDeR Models Reranking on GPT-4o CoT results k=100 (MATH+NuminaMath, all query types)													
Qwen2.5-7B-instruct	37.0	32.4	25.5	<u>41.5</u>	<u>24.9</u>	26.7	28.1	12.2	28.8	2.7	21.7	39.1	<u>26.7</u>
gte-Qwen2-7B-instruct	36.9	31.2	24.9	43.1	26.4	<u>26.3</u>	26.1	16.6	26.6	0.4	18.9	36.1	26.1

Table 9: nDCG@10 performance of rerankers on BRIGHT using questions as retrieval queries. The first-stage results are obtained by BM25 using GPT-4o CoT reasoning as queries. Results with * are taken from (Weller et al., 2025).

q_{lexical} , built from a mathematical question and its corresponding theorem in Figure 3. As illustrated in the figure, q_{CoT} integrates multiple reasoning steps to predict the relevance of a positive theorem, thereby guiding the retriever to perform complex reasoning for relevance ranking. In contrast, q_{lexical} exhibits high lexical overlap with the associated positive theorem, which helps for queries that do not require reasoning. The query q_{llmq} strikes a balance between these two extremes, as it is generated using both the mathematical question and the relevant theorem, effectively combining contextual reasoning and lexical similarity.

Examples from BRIGHT TheoremQA. We present examples from the TheoremQA theorems split of BRIGHT to qualitatively compare RaDeR with other baseline retrievers. In the example shown in Figure 4, RaDeR successfully performs reasoning to successfully retrieve the gold theorem, in contrast to a strong baseline model, Qwen2, which fails on this instance. Nonetheless, there are cases that RaDeR faces challenges. Figure 5 presents an example, where the question is about directionality in a family tree graph. Both Qwen2 and RaDeR retrievers fail to capture the query’s focus on *acyclicity*, instead retrieving results about *structural hierarchy* in the graph. This leads to in-

correct retrieval and highlights the need for further advancements in reasoning-aware retrieval models.

Analysis of Coding example from Leetcode In Figure 6, we present a qualitative example from Leetcode, which shows how recognizing mathematical substructure in a coding problem can help RaDeR retrievers. Retrieving relevant documents for this coding problem requires mathematical reasoning which RaDeR models excel in, whereas BM25 only performs lexical matching based on the word *rectangle*, leading to incorrect retrieval.

8 Conclusion

We introduce RaDeR, a suite of retrievers and rerankers designed for reasoning-intensive relevance ranking. Our approach employs a retrieval-augmented reasoning framework based on Monte Carlo Tree Search (MCTS) to generate high-quality training data. RaDeR achieves substantial improvements in both retrieval and reranking performance over state-of-the-art models on reasoning-intensive benchmarks, demonstrating strong generalizability and significantly higher data efficiency compared to existing methods.

9 Limitations

While RaDeR achieves strong performance on reasoning-intensive retrieval tasks, it has a few limitations. First, our training approach primarily focuses on examples where the retriever reasons over a single document in isolation. A promising direction for future work is to develop retrievers capable of reasoning over multiple documents jointly, where the relevance of each document is informed by the content of others, similar to the requirements in multi-hop QA tasks.

Second, RaDeR models focus on producing relevance scores without generating explicit explanations for document retrieval. Future work could explore the development of reasoning-aware retrievers that offer greater transparency and interpretability by generating explanations for their retrieval decisions, all while maintaining their efficiency for the first-stage ranking.

Lastly, we use rewards in our MCTS framework, based on the final answer of the math reasoning datasets. However, incorrect CoT reasoning path for solving a mathematical question can lead to a correct final answer, thus our training data can be noisy.

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A Detailed Discussion of Related Works

A.1 Mathematical Information Retrieval

Math Information Retrieval (Math IR) has been extensively studied within the IR community, focusing on the task of retrieving relevant mathematical documents such as theorems, formulas, similar questions, or textbooks to solve a given math problem. Early models, such as Approach0 (Zhong et al., 2021), perform retrieval by using structural similarities between the formulas in queries and documents.

MathBERT (Zhong et al., 2022) pre-trained a cross-encoder BERT-base model on a corpus of 1.69 million math documents containing both text and formulas. Several approaches (Zhong et al., 2022) explored hybrid methods combining dense neural retrievers with structural and lexical retrievers to improve retrieval performance.

The NTCIR-10 Math Pilot Task (Aizawa et al., 2013) marked one of the first collaborative efforts to establish evaluation frameworks for *mathematical formula search*, while the ARQMATH Lab tasks (Mansouri et al., 2022) extended MATH IR to a Community Question Answering (CQA) setting, utilizing user-generated data from *Math StackExchange*.

More recently, datasets such as **BRIGHT** (Su et al., 2024) and RAR-B (Xiao et al., 2024) have been introduced as evaluation benchmarks for Math IR, focusing on tasks including *Theorem Retrieval*, *Similar Questions Retrieval*, and *Answer Retrieval*. In this work, we adopt both **BRIGHT** and RAR-B as evaluation datasets to assess the effectiveness of our trained retriever.

A.2 CoT Reasoning

IRCoT (Trivedi et al., 2023) uses the last reasoning step as a retrieval query, conditioning future steps on retrieved documents. ITER-RETGEN (Shao et al., 2023) alternates between retrieval and generation, showing improvements in tasks such as multi-hop QA and fact verification. REACT (Yao et al., 2023b) iteratively generates (thought, action, observation) sequences, using intermediate reasoning to drive retrieval. Toolformer (Schick et al., 2023) employs self-supervised training to help models autonomously determine when to invoke external retrieval tools, like the *Wikipedia Search API*.

A.3 Reasoning-Based Re-Ranking Models

RANK-1 (Weller et al., 2025) employs knowledge distillation from DeepSeek-R1, extracting over 600K reasoning examples from the MS-MARCO dataset (Nguyen et al., 2017) to train a re-ranking model. Similarly, InteRank (Samarinas and Zamani, 2025) uses reinforcement learning to train a 3B-parameter re-ranking model, generating reasoning explanations alongside relevance scores for (query, document) pairs. Despite their improved retrieval quality, these re-ranking models remain inherently dependent on the first-stage retriever’s candidate set, typically derived from lexical or semantic matching, thus limiting their effectiveness on reasoning-intensive retrieval tasks. In contrast, RaDeR develops a dedicated first-stage retriever from pretrained language models, including *Qwen2.5* (Yang et al., 2024), *Llama 3.1* (Grattafiori et al., 2024), and *gte-Qwen2-7B-instruct* (Li et al., 2023b), tailored explicitly to reasoning-intensive search scenarios.

B Sampling MCTS Rollout Solutions

The MCTS proceeds with multiple iterations of four main processes: *selection*, *expansion*, *simulation* and *backpropagation*. To balance exploration and the exploitation, the *selection* step starts from the root node and uses Upper Confidence Bounds applied to trees (UCT) to traverse through child nodes, continuing until a leaf node is reached. Formally, we select the node with maximum UCT value at each branch of the traversal:

$$\text{UCT}(s, a) = \frac{Q(s, a)}{N(s, a)} + c \sqrt{\frac{\ln(N_{\text{parent}}(s))}{N(s, a)}}$$

where $N(s, a)$ is the number of times node s has been visited till now and $Q(s, a)$ is the expected reward of node s under action a and c is a hyperparameter. If the leaf node is not a *terminal* node, the *expansion* step adds child nodes to the leaf node to represent potential future actions. The *simulation* step selects one of the newly added child nodes at random and performs rollouts/simulations by selecting actions randomly until we reach a terminal node t . Based on whether the terminal node t reaches the correct gold answer G , we calculate a reward value $R(t)$ and update $Q(s, a)$ values for all the nodes s_i in the collected solution trajectory $M \oplus s_1 \oplus s_2 \oplus \dots \oplus s_t$ as:

$$Q(s_i, a) = Q(s_i, a) + R(t)$$

The $N(s, a)$ values are also incremented as

$$N(s_i, a) = N(s_i, a) + 1$$

C Math Datasets for Data Generation

To construct our training dataset, we leverage mathematical reasoning benchmarks consisting of natural language math questions paired with a correct answer, typically represented as a boxed numerical value in $\text{\LaTeX}$. Our data generation pipeline does not require access to gold step-by-step solutions.

MATH The MATH dataset comprises mathematical problems across 8 different subject types (Pre-algebra, Precalculus, Algebra, Geometry, Intermediate Algebra, Counting and Probability, and Number Theory) and five difficulty levels (from 1 to 5, where 1 denotes the easiest).

NuminaMath. NuminaMath is a large-scale collection comprising 860K competition-level math problems paired with solutions. Our MCTS leverages the OrcaMATH, AMC, AIME, Chinese K-12 Exam, Olympiad, and AOP forum splits from NuminaMath.

Parameter	Value
No of RT nodes per action (Top- k)	5
No of OST nodes added (per action)	2
No of QG nodes added (per action)	1
No of rollouts	16
Max depth	6
MCTS Exploration weight C	2
MCTS Weight Scheduler	const
LLM _{gen} temperature	0.8
LLM _{gen} top- k	40
LLM _{gen} top- p	0.95
BF16	Enabled
GPUs	2 A100s

Table 10: MCTS Parameters

D MCTS Parameters

Table 10 shows the values of hyperparameters for the MCTS algorithm.

E Statistics of Synthesized Data

Table 11 shows the number of synthesized samples from each dataset for mathematical problem solving.

Dataset (query type)	# of Samples
MATH (q_{llmq})	18,586
MATH (q_{CoT})	7,312
MATH (q_{question})	7,312
MATH (q_{lexical})	9,910
MATH (all queries)	43,120
NuminaMATH (q_{llmq})	39,639
NuminaMATH (q_{CoT})	24,280
NuminaMATH (q_{question})	10,241
NuminaMATH (q_{lexical})	4,158
NuminaMATH (all queries)	78,318
MATH+NuminaMATH (all queries)	121,438

Table 11: Statistics of synthesized data for retrieval training.

F Retriever Training Details

We append an end-of-sequence token (EOS token) to the input query or document to form the input sequence to our base LLM. Thus, the vector embedding of a query or a document (denoted as t) is computed as:

$$E_t = \text{Decoder}(t_1 t_2 \cdots t_k \langle \text{eos} \rangle)[-1],$$

where $\text{Decoder}(\cdot)$ represents the LLM model (such as Qwen or Llama), which returns the last layer token representations for each input token. We take the representation of the end-of-sequence token as the representation of the input sequence $t_1, \dots, t_k$, which can be either a query q or a document d . Relevance of d to q is computed using the cosine similarity of their corresponding dense representations E_q and E_d as:

$$s(q, d) = \cos(E_q, E_d).$$

The model is then optimized end-to-end using the InfoNCE loss:

$$\mathcal{L}(q, p, D^-) = -\log \frac{\exp(s(q, p))}{\exp(s(q, p)) + \sum_{d^- \in D^-} \exp(s(q, d^-))}, \quad (1)$$

where p denotes a document relevant to the query q (based on human annotations), while D^- is the set of negative (non-relevant) documents.

G Reranker Training Details

Our reranker model is trained as pointwise reranker. The input to the model is a concatenation of the query and a candidate document, with the model

generating a score that indicates the relevance of the document to the query (Nogueira et al., 2020).

In more detail, our model reranks a query-document pair as shown below:

$$\begin{aligned} \text{input} &= \text{query: } \{q\} \text{ document: } \{d\} \text{ <eos>} \\ s(q, d) &= \text{Linear}(\text{Decoder}(\text{input})[-1]) \end{aligned}$$

Here, $\text{Decoder}(\cdot)$ represents the LLM model (such as Qwen or Llama), which returns the last layer token representations for each input token, and $\text{Linear}(\cdot)$ is a linear projection layer that maps the final hidden state corresponding to the end-of-sequence token to a scalar relevance score. The training uses same loss, used for retriever training, with no use of in-batch negatives.

Parameter	Value
Train Group Size	12
Warmup Steps	28
Per Device Train Batch Size	2
Gradient Accumulation Steps	16
DDP Timeout	1800
Temperature	0.01
Learning Rate	1e-4
LR Scheduler Type	Linear
Number of Train Epochs	1
BF16	Enabled
GPUs	2 A100s

Table 12: Retriever Training Hyperparameters.

H Retriever/Reranker Training Hyperparameters

We used *Tevatron* (Ma et al., 2025) package for training. The hyperparameters used for finetuning retrieval and reranking models are presented in Table 12.

I Retrieval Performance on BRIGHT

In addition to nDCG performance of retrieval models, we compare our RaDeR models with baselines in terms of precision at top 10 documents in Table 13 and recall at top 10 documents in Table 14 when questions are used as retrieval queries.

J Ablation of Query type for RaDeR Rerankers

We include a comparison of RaDeR rerankers trained with all query types vs ($q_{\text{llmq}} + q_{\text{CoT}} + q_{\text{lexical}}$) in Table 15.

K Ablation Studies

Effect of query types in synthesized samples. We evaluate the impact of different query types in the synthesized retrieval data by comparing variants of our RaDeR models trained on subsets including specific query types. Table 3 summarizes the results of this ablation on BRIGHT.

We observe a consistent improvement in retrieval performance as additional query types are included during training. These results highlight the *complementary* nature of the diverse query types in our synthesized training data.

L Example Prompts

This section provides the prompts used in our work.

M Prompt for OST thought action

We provide the prompts used for our **One Step Thought Action** (OST) A_1 used in the MCTS, in Figure 7.

N Prompt for CRS action

We provide the prompts for the Complete Reasoning Steps action A_2 of MCTS, in Figure 8.

O Prompt for MCTS Query Generation

In this section, we provide the prompts for the query generation action A_3 used in our MCTS framework. These prompts guide LLM_{gen} to generate a plausible hypothetical theorem that could help solve the current subproblem. This format of the prompt is chosen to increase the likelihood of term matching retrievers finding the relevant theorems. The instruction is provided in Figure 9 and the few shot examples used are presented in Figure 10.

P Prompt for Self Reflection MCTS

In this section, we provide the prompts for the *self reflection* and *self-summarization* mechanisms used in our MCTS framework. The self-reflection prompt is shown in Figure 11 and the summarization prompt used is presented in Figure 12.

Q Prompt for LLM Reasoning Query generation

In this section, we provide the prompts used for generating $\text{LLM}_{\text{query}}$ based on the input math question, the reasoning CoT context and the theorem retrieved in the MCTS. The instruction with the few-shot examples are provided in Figure 13.

	StackExchange						Coding		Theorem-based			Avg.	
	Bio	Earth	Econ	Psy	Rob	Stack	Sus	Leet	Pony	AoPS	TheoQ		TheoT
Sparse and Open-source Baselines													
BM25*	7.6	12.4	7.1	6.0	5.6	8.0	6.1	6.0	7.9	3.1	2.2	1.3	6.1
Qwen*	13.5	14.1	8.2	11.2	5.8	10.1	6.1	6.3		9.7	7.1	6.2	7.3
GritLM*	11.1	12.7	9.2	10.8	6.8	6.0	6.8	7.5	17.9	4.6	5.7	5.3	8.7
Inst-XL*	10.0	13.8	10.6	11.0	6.7	8.8	9.2	6.3	4.6	4.7	3.3	1.9	7.6
E5*	8.9	10.2	8.1	8.6	7.2	4.6	6.9	6.9	4.9	4.2	5.7	6.5	6.9
Proprietary Baselines													
Google*	10.3	12.2	8.9	11.4	5.6	8.3	7.9	6.9	3.6	5.0	4.8	4.2	7.4
Voyage AI*	11.0	9.9	9.6	11.0	5.6	7.3	7.5	7.5	1.1	5.0	5.6	3.2	7.0
RaDeR Models (MATH dataset, $q_{llmq} + q_{CoT} + q_{lexical}$)													
Qwen2.5-7B-instruct	11.6	11.7	6.8	9.9	5.3	7.9	6.7	8.5	8.0	8.2	10.2	9.9	8.7
RaDeR Models (MATH+NuminaMath datasets, all query types)													
Qwen2.5-7B-instruct	12.2	11.5	7.8	10.6	5.7	7.4	6.8	8.1	6.2	6.0	9.2	11.3	8.6

Table 13: Precision@10 performance using question as queries for retrieval over BRIGHT.

	StackExchange							Coding		Theorem-based			Avg.
	Bio	Earth	Econ	Psy	Rob	Stack	Sus	Leet	Pony	AoPS	TheoQ	TheoT	
Sparse and Open-source Baselines													
BM25*	21.8	31.4	16.8	15.5	19.4	16.8	21.1	29.5	3.6	6.0	11.4	9.0	16.9
Qwen*	38.2	40.6	18.5	29.5	14.5	22.4	17.4	32.1	4.6	14.8	30.0	39.4	25.2
GritLM*	30.3	38.8	18.3	26.9	21.3	15.1	23.4	36.3	8.2	9.4	26.2	26.6	23.4
Inst-XL*	27.3	38	25.4	35.6	22	21.1	23.9	31.8	2.5	8.9	16.6	9.8	21.9
E5*	22.0	29.4	18.4	18.3	18.7	11.9	23.0	34.6	2.4	8.2	27.2	34.8	20.7
Proprietary Baselines													
Google*	26.1	36.9	20.6	31.4	17.7	21.6	23.7	33.5	1.9	10.4	24.0	22.1	22.5
Voyage AI*	29.3	31.2	21.0	31.0	15.0	17.5	20.5	41.5	0.6	8.7	28.5	15.4	21.7
RaDeR Models (MATH dataset, $q_{llmq} + q_{CoT} + q_{lexical}$)													
Qwen2.5-7B-instruct	30.6	36.9	19.5	30.1	18.5	28.6	21.1	46.5	6.4	18.1	48.3	56.0	30.1
gte-Qwen2-7B	41.6	42.5	26.1	41.2	18.6	32.5	31.7	44.7	3.4	13.4	32.5	47.9	31.3
Llama3.1-8B-Instruct	35.7	33.5	18.4	36.4	14.8	28.5	20.3	45.2	6.4	7.7	37.0	43.4	27.3
RaDeR Models (MATH+NuminaMath datasets, all query types)													
Qwen2.5-7B-instruct	31.3	34.2	21.5	31.9	18.8	30.5	21.5	45.6	5.1	11.7	44.7	63.8	30.1

Table 14: Recall@10 performance using question as queries for retrieval over BRIGHT.

Methods	StackExchange							Coding		Theorem-based			Avg.
	Bio	Earth	Econ	Psy	Rob	Stack	Sus	Leet	Pony	AoPS	TheoQ	TheoT	
First stage: BM25 with question as query, k=100													
$\mathbf{q}_{llmq} + \mathbf{q}_{CoT} + \mathbf{q}_{lexical}$	24.5	28.5	14.1	24.7	13.0	14.0	19.6	18.9	16.4	4.7	20.9	13.7	17.8
$\mathbf{q}_{llmq} + \mathbf{q}_{CoT} + \mathbf{q}_{lexical} + \mathbf{q}_{question}$	26.9	30.6	17.0	24.9	18.2	17.8	20.5	23.7	14.3	4.4	22.4	13.6	19.5
First stage: BM25 with GPT4-o CoT as query, k=100													
$\mathbf{q}_{llmq} + \mathbf{q}_{CoT} + \mathbf{q}_{lexical}$	37.3	29.1	21.4	39.2	17.5	21.1	26.6	11.7	32.0	1.2	20.9	35.3	24.4
$\mathbf{q}_{llmq} + \mathbf{q}_{CoT} + \mathbf{q}_{lexical} + \mathbf{q}_{question}$	40.8	31.8	25.0	39.7	21.6	25.7	27.2	17.3	29.9	1.6	22.4	36.9	26.7

Table 15: Comparison of Reranker performance with two different query types on BRIGHT under two first-stage retrieval settings . Model used for all settings is Qwen2.5-7B-instruct. Each number reports nDCG@10.

Example
Question (M): Let $f(x)$, $g(x)$, and $h(x)$ be linear functions. Define $j(x) = \max\{f(x), g(x), h(x)\}, \quad k(x) = \min\{f(x), g(x), h(x)\}.$ Suppose the graph of $y = j(x)$ over $-3.5 \leq x \leq 3.5$ consists of three connected line segments: it increases linearly to $(-2, 2)$, remains constant at $y = 2$ from $x = -2$ to $x = 2$, and then increases linearly from $(2, 2)$ to $x = 3.5$. Let ℓ denote the total length of the graph of $y = k(x)$ over the same interval. Find ℓ^2. Positive document (p): Linear Real Function - A real function f is called linear if it has the form $f(x) = ax + b$, where a and b are real numbers.
Different Query Types
LLM generated query (q_{llmq}): Calculate the length of the graph of a piecewise linear function defined by the minimum of three linear functions over a given interval. CoT reasoning query (q_{CoT}): Question + To determine the length of the graph of $y = k(x)$ for $-3.5 \leq x \leq 3.5$, we first need to understand the behavior of $j(x)$, the maximum of the three linear functions $f(x)$, $g(x)$, and $h(x)$. From the graph, we can see that $j(x)$ transitions between the three linear functions at points where two of the functions are equal. The graph of $j(x)$ has horizontal segments and linear segments connecting the points where the maximum changes. Lexical query (q_{lexical}): What is the definition of a linear real function and what form must it take for all real numbers x?

Figure 3: Examples of different query types from our retrieval training dataset built for the given math question.

Example
Query: Everyone that you invite to a party will be either a fan of <i>football</i> or a fan of <i>basketball</i>, but never both. What is the smallest number of guests you need to invite to ensure that there are either 3 people who are all fans of football or 3 people who are all fans of basketball? Gold Theorem Id: 7627 Gold Theorem: <i>Ramsey Theorem: Ramsey's Theorem guarantees that for any edge-coloring of a sufficiently large complete graph, there exists a monochromatic complete subgraph. More formally, for integers $n_1, n_2, \dots, n_c$, there exists a Ramsey number $R(n_1, \dots, n_c)$ such that any c-coloring of a complete graph on $R(n_1, \dots, n_c)$ vertices contains a monochromatic K_{n_i} in some color i. (Note that the gold theorem has no lexical overlap with the original question.)</i>
Qwen2 retriever
Top Retrieved Theorem: <i>Pigeonhole Principle: Let S be a finite set with n elements, partitioned into k subsets $S_1, S_2, \dots, S_k$. Then at least one subset S_i satisfies:</i> $ S_i \geq \left\lceil \frac{n}{k} \right\rceil$ (Incorrect retrieved theorem)
RaDeR GTE-Qwen2-7B-instruct (all query types) retriever
Top Retrieved Theorem: <i>Ramsey's Theorem guarantees that for any edge-coloring of a sufficiently large complete graph, there exists a monochromatic complete subgraph. More formally, for integers $n_1, n_2, \dots, n_c$, there exists a Ramsey number $R(n_1, \dots, n_c)$ such that any c-coloring of a complete graph on $R(n_1, \dots, n_c)$ vertices contains a monochromatic K_{n_i} in some color i. (Correct theorem retrieved)</i>
Explanation
Explanation: While the Pigeonhole Principle might appear relevant at first glance due to the presence of element selection in the query, the problem is more appropriately modeled as a graph coloring task. Specifically, it reduces to a two-coloring of the edges of a complete graph, where the objective is to guarantee the existence of a monochromatic triangle. Our RaDeR retriever demonstrates the ability to perform such nuanced reasoning, correctly interpreting the structural semantics of the query and retrieving the correct gold document: the Ramsey Theorem.

Figure 4: Example of RaDeR success case compared to Qwen2, from TheoremQA theorems of BRIGHT.

Example
Query: Imagine you have a family tree that shows the lineage from ancestors to their descendants, with arrows pointing from parents to children. Can this family tree, with its directed lineage paths, be accurately represented without the arrows while still maintaining the correct relationships? True or false? Gold Theorem Id: 3778 Gold Theorem: <i>The "girth" of G is the smallest length of any cycle in G. An acyclic graph is defined as having a girth of infinity. (Note that the gold theorem has no lexical overlap with the original question.)</i>
Qwen2 retriever
Top Retrieved Theorem: <i>Rooted Tree Corresponds to Arborescence: Let $T = (V, E)$ be a rooted tree with root r. Then there exists a unique orientation of T that forms an r-arborescence.</i> (Incorrect theorem retrieved)
RaDeR GTE-Qwen2-7B-instruct (all query types) retriever
Top Retrieved Theorem: <i>Rooted Tree Corresponds to Arborescence: Let $T = (V, E)$ be a rooted tree with root r. Then there exists a unique orientation of T that forms an r-arborescence.</i> (Incorrect theorem retrieved)
Explanation
Explanation: This case demonstrates a semantic mismatch: while the retrieved theorem on arborescences aligns superficially with the query's mention of family trees, the actual focus is on acyclicity, better captured by the gold theorem on girth. The error highlights how surface-level similarity can mislead retrieval.

Figure 5: Example of RaDeR Failure cases from TheoremQA theorems BRIGHT

Example
Query: Problem: Given an array of integers heights representing the histogram's bar heights (each bar has width 1), return the area of the largest rectangle in the histogram. Example 1. Input: heights = [2,1,5,6,2,3]. Output: 10. The largest rectangle has area 10. Example 2. Input: heights = [2,4]. Output: 4. Constraints. $1 \leq \text{heights.length} \leq 10^5, 0 \leq \text{heights}[i] \leq 10^4$.
BM25 retriever
Top Retrieved Document: Problem: Given an array rectangles, where $\text{rectangles}[i] = [\text{width}_i, \text{height}_i]$, return the number of pairs (i, j) with $i < j$ such that the rectangles have the same width-to-height ratio, i.e., $\frac{\text{width}_i}{\text{height}_i} = \frac{\text{width}_j}{\text{height}_j}$ (using decimal division). (Incorrect retrieved theorem)
RaDeR Llama3.1-8B-instruct retriever
Top Retrieved Document: Problem: Given a binary matrix matrix of size rows $\times$ cols, return the area of the largest rectangle containing only '1's. Approach: For each row, treat it as the base of a histogram where the height at each column counts consecutive '1's up to that row. For each histogram, compute the largest rectangle using a stack-based approach (<i>similar to the Largest Rectangle in Histogram problem</i>). Constraints. $1 \leq \text{rows}, \text{cols} \leq 200; \text{matrix}[i][j] \in \{'0', '1'\}$. (Correct theorem retrieved)
Explanation
The coding problem has a mathematical substructure because it requires computing the maximum rectangular area under a histogram, which involves geometric reasoning and optimization over intervals of the graph. Such a structure helps RaDeR to do better at code retrieval tasks.

Figure 6: Analysis of RaDeR Llama3.1-8B-instruct success case for Coding Example from *Leetcode* split of BRIGHT

Instruction
A chat between a curious user and an AI assistant. The assistant gives step-by-step solutions to the user's questions. At the final step, a conclusive answer is given in the format of "The answer is: boxed{ANSWER}.", where ANSWER should be a numeric answer. <i>{few shot examples}</i> Question: {question} Response: Let's think step by step.
Few Shot Examples
Question: Gracie and Joe are choosing numbers on the complex plane. Joe chooses the point $1 + 2i$. Gracie chooses $-1 + i$. How far apart are Gracie and Joe's points? Response: Let's think step by step. Step 1: Distance formula is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Step 2: Joe's point $(1, 2)$, Gracie's $(-1, 1)$. Step 3: Distance $\sqrt{((-1) - (1))^2 + ((1) - (2))^2} = \sqrt{4 + 1} = \sqrt{5}$. Step 4: Points are $\boxed{\sqrt{5}}$ units apart. Step 5: The answer is: $\sqrt{5}$. Question: What is the sum of all positive integer values of n for which $\frac{n+6}{n}$ is an integer? Response: Let's think step by step. Step 1: $n + 6$ divisible by n. Step 2: n must be factor of 6. Step 3: Factors of 6 are 1, 2, 3, 6. Sum is $1 + 2 + 3 + 6 = \boxed{12}$. Step 4: The answer is: 12. Question: Abigail, Beatrice, and Carson sell eggs in cartons of 12. Abigail has 37, Beatrice 49, Carson 14 eggs. How many eggs remain after selling all cartons? Response: Let's think step by step. Step 1: Total eggs $37 + 49 + 14 = 100$. Step 2: Divide by 12: $100 \div 12 = 8$ cartons, remainder 4. Step 3: Remaining eggs $\boxed{4}$. Step 4: The answer is: 4. Question: Circle T has center $T(-2, 6)$, reflected across y-axis, translated 8 units down. Find coordinates of image center. Response: Let's think step by step. Step 1: Reflect across y-axis: $(-(-2), 6) = (2, 6)$. Step 2: Translate down 8 units: $(2, 6 - 8) = (2, -2)$. Step 3: Image coordinates $\boxed{(2, -2)}$. Step 4: The answer is: $(2, -2)$.

Figure 7: Prompt for MCTS One Step Thought Action

Instruction
You are given context about a mathematical question. Your job is to generate the next steps of the solution and complete the solution. In the end of your response, a final answer is given in the format of "\$\boxed{<ANSWER>}\$" , where <ANSWER> should be a numeric result or a math expression.
Few Shot Examples
Context: Gracie and Joe are choosing numbers on the complex plane. Joe chooses the point $1 + 2i$. Gracie chooses $-1 + i$. How far apart are Gracie and Joe's points? The distance between two points (x_1, y_1) and (x_2, y_2) in the complex plane is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Joe's point is $(1, 2)$ and Gracie's point is $(-1, 1)$. Next steps: The distance is $\sqrt{((-1) - (1))^2 + ((1) - (2))^2} = \sqrt{4 + 1} = \sqrt{5}$. Therefore, Gracie and Joe's points are $\boxed{\sqrt{5}}$ units apart.
Context: What is the sum of all positive integer values of n for which $\frac{n+6}{n}$ is an integer? Next steps: We want $\frac{n+6}{n}$ to be integer, thus $n + 6$ divisible by n. Since n positive, n must factor 6. Factors: 1, 2, 3, 6. Sum is $1 + 2 + 3 + 6 = \boxed{12}$.
Context: Abigail, Beatrice, and Carson sell eggs in cartons of 12. Abigail has 37 eggs, Beatrice has 49, Carson has 14. First, total eggs: $37 + 49 + 14 = 100$. Divide by 12: $100 \div 12 = 8$ remainder 4. Next steps: Eggs remaining: $\boxed{4}$. The answer is: 4.
Context: Circle T has center $T(-2, 6)$, reflected across y-axis, translated 8 units down. Reflecting across y-axis negates x-coordinate. Next steps: Reflection gives $(2, 6)$. Translating down 8 units: $(2, 6 - 8) = (2, -2)$. Therefore, coordinates are $\boxed{(2, -2)}$.

Figure 8: Prompt for Complete Remaining Steps (CRS) Action

Instruction
You are given a mathematical question and an intermediate solution. What are the mathematical concepts, theorems, formulas that would be useful for solving this question. Please provide the theorem name, followed by the theorem statement, followed by the preconditions in the theorem, and why the preconditions are satisfied in the question we have. Also mention which specific subjects in math this theorem corresponds to. List out as many number of theorems that are highly relevant to this question. Do not output the final solution. Do not generate theorems which are already present in the intermediate solution.

Figure 9: Instruction for MCTS Query Generation Action

Few Shot Examples

Theorem: Polynomial Division Algorithm

Theorem Statement: For any two polynomials $P(z)$ (dividend) and $D(z)$ (divisor), with $\deg(P(z)) \geq \deg(D(z))$, there exist unique polynomials $Q(z)$ (quotient) and $R(z)$ (remainder) such that: $P(z) = D(z)Q(z) + R(z)$.

Question: Find quotient of $\frac{3z^4-4z^3+5z^2-11z+2}{2+3z}$.

Intermediate Solution: Apply polynomial long division.

Query: (Polynomial Division) $P(z) = D(z)Q(z) + R(z)$ if $\deg(P) \geq \deg(D)$.

Preconditions Met: (1) $D(z) = 3z + 2 \neq 0$, (2) $\deg(P) = 4 > \deg(D) = 1$.

Subject: Algebra.

Theorem: Principle of Inclusion-Exclusion

Theorem Statement: For any two finite sets A and B , the size of their union is given by: $|A \cup B| = |A| + |B| - |A \cap B|$.

Question: Probability of palindrome (letters/digits) in plates, simplified as $\frac{m}{n}$, find $m + n$.

Intermediate Solution: Compute separately, combine using inclusion-exclusion.

Query: (Inclusion-Exclusion) $|A \cup B| = |A| + |B| - |A \cap B|$.

Preconditions Met: (1) Sets finite (letters/digits), potential overlap possible.

Subject: Combinatorics.

Theorem: Basic Multiplication Principle

Theorem Statement: If there are m ways to do something and n ways to do another thing, then there are $m \cdot n$ to do both things.

Question: Pages written/year if 3-page letters to 2 friends twice weekly?

Intermediate Solution: Pages/week calculation, then yearly total.

Query: (Multiplication Principle) Actions with m and n ways yield $m \times n$ combined ways.

Preconditions Met: Counts defined clearly; independent actions.

Subject: Arithmetic.

Figure 10: Few shot examples for MCTS Query Generation Action

Instruction
You are given a mathematical question, an intermediate solution and a mathematical theorem which was retrieved denoted as Retrieved Document. First, please judge whether the mathematical theorem is relevant with the question and the intermediate solution, and put it in the relevant field. If the provided content is irrelevant to the question and the context, explain the reason in the relevant reason field. The format will be as follows: Question: [question] Intermediate solution: [intermediate solution] Retrieved Document: [theorem] Relevant: [relevance label] Reason: [reason].
Few Shot Examples
Question: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous, with $f(0) = 0$ and $f(x + y) = f(x) + f(y) + xy$. Find degree of f. Intermediate Solution: Define $g(x) = f(x) - \frac{x^2}{2}$. Retrieved Document: Cauchy's equation $f(x + y) = f(x) + f(y)$, continuous solution linear: $f(x) = cx$. Relevant: True Reason: Defining $g(x)$ transforms into Cauchy's form, yielding $g(x) = cx$ and thus $f(x) = \frac{x^2}{2} + cx$ degree 2. Question: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ differentiable with $f'(x) = f(x) + x$. Find $f(x)$. Intermediate Solution: Requires solving differential equation directly. Retrieved Document: Rolle's Theorem guarantees $f'(c) = 0$ under certain continuity/differentiability conditions. Relevant: False Reason: There is no direct application of Rolle's Theorem to the differential equation provided, as the theorem does not help in finding the solution to the equation $f'(x) = f(x) + x$. Question: Battery lifetime exponential mean 10 hours. Probability battery lasts at least 15 hours? Intermediate Solution: Use exponential distribution's CDF. Retrieved Document: Markov's Inequality provides upper bound $P(X \geq a) \leq \frac{E[X]}{a}$. Relevant: False Reason: Markov's gives bounds, not exact probabilities; exact CDF calculation is necessary here.

Figure 11: Prompt for Self Reflection MCTS

Instruction
Given the statement of a mathematical theorem in a structured latex format, convert it to a simpler natural language format by removing latex notations.
Few Shot Examples
Input ProofWiki Theorem (in latex): <i><Latex Section></i>: Quadratic Irrational is Root of Quadratic Equation <i><Tags></i>: Algebra, Quadratic Equations, Quadratic Irrationals <i><begin theorem></i> Let x be a quadratic irrational. Then x is a solution to a quadratic equation with rational coefficients. Generated Natural language theorem: Quadratic Irrational is Root of Quadratic Equation - A quadratic irrational number is always the root of some quadratic equation with rational coefficients.

Figure 12: Prompt for Self Summarization of Retrieved theorems

Instruction
Given a math question, its partial solution (may be empty), and a retrieved theorem, do the following: Identify the preconditions of the theorem and explain why they hold in the given question. Using these preconditions, generate a general retrieval query that captures the key mathematical idea needed in the partial solution. The query should be a single sentence.
Few Shot Examples
Question: A warehouse needs to store 65 identical boxes using a set of identical shelves, each of which can hold up to 8 boxes. What is the minimum number of shelves required to store all the boxes? Partial solution: To determine the minimum number of shelves required, we divide the total number of boxes by the capacity of each shelf. Since the number of shelves must be a whole number, we round up to 9 shelves. Theorem: If n items are put into m containers, with $n > m$, then at least one container must contain more than one item. Preconditions: (1) There are more items than containers. (2) The items are distributed into containers. Why Preconditions are Satisfied: (1) The warehouse has 65 boxes (items) and needs to distribute them among shelves (containers), where each shelf can hold up to 8 boxes. (2) Since 65 is greater than 8, multiple boxes must be placed on each shelf to store all of them. Generated Query: Minimizing the number of boxes needed to store a given number of objects with fixed capacity constraints.

Figure 13: Prompt for LLM generated query (q_{llmq}) generation

Resource	URL
Codebases	
Tevatron	https://github.com/texttron/tevatron
rStar	https://github.com/zhentingqi/rStar
BRIGHT	https://github.com/xlang-ai/BRIGHT
Models	
Qwen2.5 models	https://huggingface.co/Qwen
Llama3.1-8B-instruct	https://huggingface.co/meta-llama
gte-Qwen2-7B-instruct	https://huggingface.co/Alibaba-NLP
Datasets	
BRIGHT dataset	https://huggingface.co/datasets/xlangai/BRIGHT
RAR-B dataset	https://huggingface.co/RAR-b
MSMARCO dataset	https://ir-datasets.com/msmarco-passage.html

Table 16: Online Tools/Resources used for RaDeR