A Logical Analysis of Autosegmental Approaches to Root-and-Pattern Morphology in Arabic

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Abstract

Autosegmental approaches to Arabic root-and-pattern morphology generally take a three-tier approach, with tiers corresponding to the prosodic template, consonantal root, and affixes (e.g., McCarthy, 1981); association between these tiers proceeds from left-to-right. However, Jardine (2017) shows that left-to-right association exceeds regular computation for autosegmental representations of arbitrary length, challenging the cognitive plausibility of this approach. This paper demonstrates that in the case of Arabic morphology, the constraints of the system itself — in particular, the finite length of the consonantal root — allow such a left-to-right autosegmental association to not only be definable with Monadic Second Order (MSO) logic, but with First Order logic. This paper introduces a logical relational structure formalizing the three-tier autosegmental representations and defines a set of transductions which apply in parallel over these structures to yield well-formed root and affix associations.

1 Introduction

Semitic languages, including Arabic, are known for their derivational "root-and-pattern" morphology in which a single, typically triconsonantal, root is associated with various prosodic patterns to yield semantically related stems. Previous work has accounted for root-and-pattern morphology in the framework of Autosegmental Phonology (Goldsmith, 1976; Clements and Ford, 1979; McCarthy, 1981; Clements and Keyser, 1983, i.a.). Under such accounts, root consonants are taken as the autosegmental melody to be associated to the melody-bearing consonantal elements of the prosodic template with a left-to-right association. This left-to-right association is evidenced by forms such as *KTaaBaB* from the consonantal root *KTB*, in which the final consonant of the root fills both the third and fourth position in the prosodic template.

A great deal of work has demonstrated that patterns in natural language belong to the class of *regular* languages or some *subregular* subset of these (Chomsky, 1957, 1963; Kaplan and Kay, 1994; Heinz, 2018; Graf, 2022, i.a.). The regular class of languages is known to be equal in expressive power to Monadic Second-Order logic (MSO) on finite words; thus, we expect natural language patterns to be adequately captured by MSO logic (Büchi, 1960). However, Jardine (2017) shows that left-to-right association of autosegmental representations of arbitrary length is not definable with MSO logic, and thus is not regular. This computational complexity challenges the cognitive plausibility of the autosegmental approach, particularly when contrasted with the subregular nature of other morphophonological patterns.

In this paper, I demonstrate that the constraints of the Arabic root-and-pattern system — in particular, the finite length of the root — allow us to define transductions over autosegmental representations using only First Order logic, a subset of MSO. The present work focuses only on triconsonantal roots, but the analysis can and should be expanded for roots of other lengths (§4). This paper proceeds as follows: in §2, I introduce the Arabic data and the autosegmental analysis of McCarthy (1981). In §3.1, I define the relational model of autosegmental representations which we will use in our transductions, and in §3.2 I define the set of logical transductions used to capture the Arabic derivational morphological system as analyzed by McCarthy. Finally, §3.3 illustrates these representations and transductions with a concrete example, §4 discusses these results, and §5 concludes.

2 Root & Pattern Morphology in Arabic

Table 1 gives the Modern Standard Arabic data analyzed by McCarthy (1981): the fifteen *binyanim* of the root *KTB*, where *binyanim* is a term bor-

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Binyan	Perfective Active	Perfective Passive	Imperfective Active	Imperfective Passive	Participle Active	Participle Passive
I	KaTaB	KuTiB	aKTuB	uKTaB	KaaTiB	maKTuuB
II	KaTTaB	KuTTiB	uKaTTiB	uKaTTaB	muKaTTiB	muKaTTaB
III	KaaTaB	KuuTiB	uKaaTiB	uKaaTaB	muKaaTiB	muKaaTaB
IV	ʔaKTaB	ʔuKTiB	uʔaKTiB	uʔaKTaB	muʔaKTiB	muʔaKTaB
V	taKaTTaB	tuKuTTiB	ataKaTTaB	utaKaTTaB	mutaKaTTiB	mutaKaTTaB
VI	taKaaTaB	tuKuuTiB	ataKaaTaB	utaKaaTaB	mutaKaaTiB	mutaKaaTaB
VII	nKaTaB	nKuTiB	anKaTiB	unKaTaB	munKaTiB	munKaTaB
VIII	KtaTaB	KtuTiB	aKtaTiB	uKtaTaB	muKtaTiB	muKtaTaB
IX	KTaBaB		aKTaBiB		muKTaBiB	
X	staKTaB	stuKTiB	astaKTiB	ustaKTaB	mustaKTiB	mustaKTaB
XI	KTaaBaB		aKTaaBiB		muKTaaBiB	
XII	KTawTaB		aKTawTiB		muKTawTiB	
XIII	KTawwaB		aKTawwiB		muKTawwiB	
XIV	KTanBaB		aKTanBiB		muKTanBiB	
XV	KTanBay		aKTanBiy		muKTanBiy	

Table 1: The root *KTB* "write" in all fifteen binyanim, taken from McCarthy (1981); gaps indicate binyanim that are regularly intransitive and thus not susceptible to passivization. Root consonants are indicated with capitalization.

rowed by McCarthy from Hebrew morphology for the classification of derived stems from a given root. Throughout the paper, root consonants are capitalized to clearly separate them from affixes.

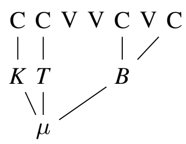
Under McCarthy's analysis, which is couched in the framework of Autosegmental Phonology (Goldsmith, 1976; Clements and Ford, 1979; Clements and Keyser, 1983), each form in Table 1 is derived by associating the given triconsonantal root with a melody of consonantal slots and vowels corresponding to the given binyan, as well as affixes in some cases. The prosodic template of a given binyan corresponds to the *segmental* level of the autosegmental analysis, while the root composes the *melody* on a single, morphologically-defined tier. McCarthy analyzes affixes — consonant(s) appearing in a given binyan which are not a part of the root — on a separate autosegmental tier, specified for the same features as the root consonants. The role of the phonological apparatus under this analysis, then, is to take in underlying unassociated elements of these three tiers and output a surface form with well-formed associations between them.

McCarthy defines this prosodic template solely in terms of the features ±Segmental and ±Syllabic, with the relevant combinations being abbreviated as C ([+Segmental, -Syllabic]) and V ([+Segmental, +Syllabic]). I do not make a commitment as to whether vowel features are represented in the prosodic template or separately, as it would have little bearing on the analysis I present in §3.2; see McCarthy for discussion. Under McCarthy's analysis, the consonantal root takes as melody-bearing elements the [-Syllabic] positions of the prosodic template of the given binyan.

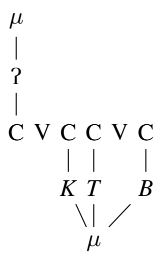
The consonants of the root and any affixes are defined in terms of all relevant phonological features except ±Segmental and ±Syllabic, though they are often represented as segments for ease of exposition. For example, the *K* in (1) indicates the bundle {Dorsal, -Voice, -Continuant, -Sonorant, ...} save any specification for Segmental or Syllabic.

McCarthy assumes that material on the affixal tier is associated with the prosodic template *before* material on the root tier, so that the root tier material may then be associated from left-to-right with the remaining slots on the prosodic template, following the universal constraints of Clements and Ford (1979). This left-to-right association is evidenced in forms such as *KTaaBaB*, as shown in (1). To illustrate the affixation process, the proposed autosegmental representation of *ʔaKTaB*, which contains the prefix [ʔ-], is given in (2) .

(1) The autosegmental representation of *KTaaBaB* proposed by McCarthy.

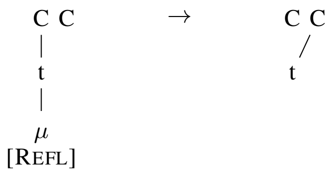


(2) The autosegmental representation of *ʔaKTaB* proposed by McCarthy.



It is generally sufficient to associate the affix with the first consonant on the prosodic template, following the universal left-right association proposed by Clements and Ford. The reflexive [t], however, is prefixed in binyan V (***t-****aKaTTaB*) and VI (***t-****aKaaTaB*), but infixed in binyan VIII (*K-****t-****aTaB*). To account for this, McCarthy proposes a "flop" rule which switches the association of the affix to the second slot in binyan VIII (3). This rule applies after the association of the affix, but before the association of the root, so that the root can associate left-to-right with the remaining available slots. For example, *KtaTaB* is derived by first associating the affix [t] with the CCVCVC template and applying the flop rule, so that when the root *KTB* is associated, *K* will fill the empty first consonant slot, *T* will fill the next empty consonant slot, which is the third, and *B* will fill the final slot.

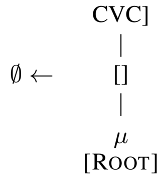
(3) McCarthy's binyan VIII flop rule.



In addition to [t], [-y] in binyan XV is purely suffixing, while [-n-] and [-w-] in binyanim XII-XV are infixing. Because they are *never* prefixed, McCarthy argues that there is no reason to posit a flop rule of the form of (3) for these binyanim, but rather that the rules of association themselves must indicate where these affixes are to be associated to the prosodic template. It is straightforward to see that this approach could also be applied to binyan VIII rather than the flop rule, and this is what we will do in §3.2.1. Though one could argue that such an analysis of binyan VIII loses the insight that [t-] is a prefix in the other two binyanim in which it appears, either method requires a binyan-specific rule, whether we only apply (3) in binyan VIII or only associate [t-] directly with the second slot in this binyan. This insight is crucial not only for the transductions we define in §3.2.1, but also for the representations in §3.1.

Finally, McCarthy addresses the gemination of the medial root consonant in binyanim II, V, and XIII (e.g., *KaTTaB*), with an "erasure" rule (4) which erases the association between the final root consonant and the penultimate consonantal slot; the penultimate C is then re-associated with the autosegement associated with the slot to its left. Erasure is also a *binyan-specific* rule.

(4) McCarthy's erasure rule.



Having reviewed the autosegmental analysis of McCarthy (1981), we now turn to a formalization of this approach in terms of logical transductions. Similarly to the phonological apparatus under the autosegmental approach, these transductions take in unassociated roots, prosodic templates, and affixes, and return a surface form with well-formed associations between elements on these three tiers.

3 Logical Formalization

3.1 Representations

Before defining our logical transductions, we must first define the representations that they will operate over. I build on the 2-tier relational model for autosegmental representations proposed by Jardine (2024) to give a relational model with three tiers: (1) the prosodic template, (2) affixes, and (3) the consonantal root. While Jardine (2024) uses the successor ($\triangleleft$) relation, I use the precedence ($<$) relation, allowing us to reference only the order of consonantal slots in the prosodic template (§3.2) and to capture the fact that although Arabic derivational morphology preserves the overall *order* of the consonants in the root, it does not preserve strict adjacency.

3.1.1 Model Signature

Our model $\mathfrak{R}$ is similar to the precedence string model, with the addition of two further relations: $\circ_a$ defines the association between affixes and the prosodic template, and $\circ_r$ defines the association between the root and the prosodic template:

$$\mathfrak{R} = \{<, \circ_a, \circ_r\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{R} \cup \mathcal{M} \qquad (1)$$

$\mathcal{P}$ is the set of features (unary relations) relevant for defining the prosodic template, $\mathcal{R}$ the set relevant to the roots, $\mathcal{M}$ the set relevant to the morphological affixes, and $\mathcal{C}$ the set of phonological features for consonants on both the root and affix tiers.

Following McCarthy (1981), $\mathcal{P}$ contains the features `segm` (corresponding to McCarthy's ±SEGMENTAL) and `syll` (corresponding to McCarthy's ±SYLLABIC). It also includes unary features for each of the 15 binyanim (i.e., I, II, etc); because processes such as the flop (3) and erasure (4)

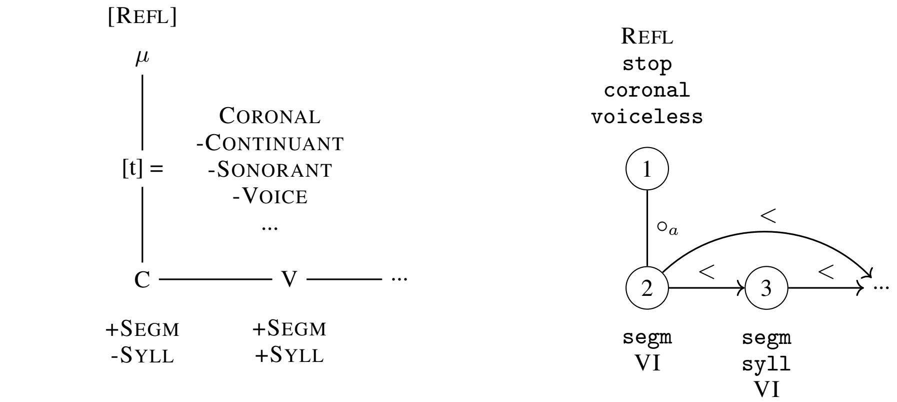


Figure 1: The original McCarthy autosegmental representation of [t-] prefix attached to the binyan VI (left) and our proposed logical representation of the same prefix on the same binyan (right)

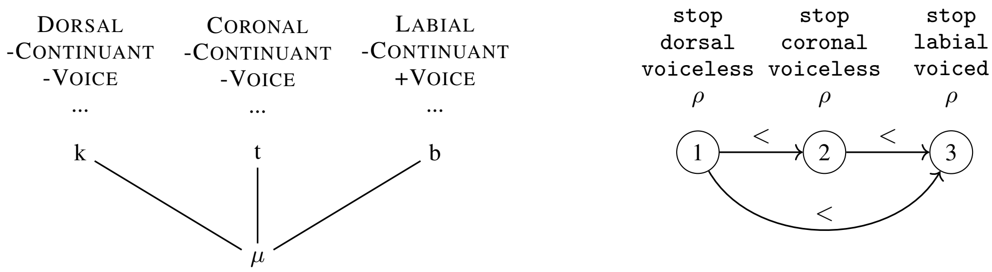


Figure 2: The original McCarthy autosegmental representation of the root *KTB* (left) and our proposed logical representation of the same root (right).

rules apply only to some binyanim, we must know which binyan we are forming to know whether the corresponding transductions (§3.2) apply. If we wished for vowels to be fully-specified in the prosodic template, we could include features such as `High`, `Round`, etc. in $\mathcal{P}$. However, whether or not these features are included has no bearing on the transductions we define in §3.2, so I do not make any theoretical claims regarding this issue.

$\mathcal{C}$ contains phonological features for consonants on the root and affix tiers, including information about place (e.g., `dorsal`), manner (e.g., `stop`) and voicing (e.g., `voiceless`), while $\mathcal{M}$ and $\mathcal{R}$ contain features relevant for specifying elements on the affix or root tiers, respectively, and differentiating between them. Under this approach, elements are on the affixal tier if they bear some feature in $\mathcal{M}$, and on the root tier if they bear some feature in $\mathcal{R}$; this aids in defining constraints on valid representations in §3.1.2.

$\mathcal{M}$ contains the morphosyntactic features (e.g., `Reflexive`) for affixes. Though McCarthy often uses μ to indicate elements on both tiers, he adds morphosyntactic features such as REFLEXIVE in his representation of the [t-] affix in the flop rule (3) to define affixal elements. While McCarthy represents features such as REFLEXIVE and μ on yet *another* autosegmental tier, I collapse these to one tier; this approach is illustrated in Figure 1 for the [t-] prefix in binyan VI. $\mathcal{R}$ contains a feature ρ which holds only for elements of morphological roots; this is analagous to the ROOT feature used by McCarthy in his definition of the erasure rule (4). The logical representation of the root *KTB* is given in Figure 2.

3.1.2 Constraints on Valid Representations

Having given the model signature of our three-tier autosegmental model, we wish to define which $\mathfrak{R}$-structures conforming to this signature are valid autosegmental representations. We begin by defining constraints on well-formed tiers. Firstly, elements in the prosodic template must bear only features which are in $\mathcal{P}$, so bearing some feature in $\mathcal{P}$ precludes bearing any features in $\mathcal{M}, \mathcal{C}$ or $\mathcal{R}$:

$$\texttt{sep}_{\texttt{p}} \overset{\text{def}}{=} p(x) \rightarrow (\forall c \in \mathcal{C})[\neg c(x)] \wedge (\forall m \in \mathcal{M})[\neg m(x)] \wedge (\forall r \in \mathcal{R})[\neg r(x)] \quad (2)$$

Similarly, segments on the affix tier and root tier must bear both consonantal features and *either* root or affix features, with one precluding the other:

$$\texttt{sep}_{\texttt{m}} \overset{\text{def}}{=} m(x) \rightarrow (\exists c \in \mathcal{C})[c(x)] \wedge (\forall p \in \mathcal{P})[\neg p(x)] \wedge (\forall r \in \mathcal{R})[\neg r(x)] \quad (3)$$

$$\texttt{sep}_\texttt{r} \overset{\text{def}}{=} r(x) \rightarrow (\exists c \in \mathcal{C})[c(x)] \wedge (\forall p \in \mathcal{P})[\neg p(x)] \wedge (\forall m \in \mathcal{M})[\neg m(x)] \tag{4}$$

We can formalize this separation of affix, root, and prosodic template elements as follows:

$$\texttt{sep} \overset{\text{def}}{=} \bigwedge_{r\in\mathcal{R},p\in\mathcal{P},m\in\mathcal{M}} \texttt{sep}_\texttt{p} \wedge \texttt{sep}_\texttt{m} \wedge \texttt{sep}_\texttt{r} \tag{5}$$

We now define whether an element is a prosodic template slot, affixal segment, or root segment:

$$\texttt{p\_slot}(x) \overset{\text{def}}{=} \bigvee_{p\in\mathcal{P}} p(x) \tag{6}$$

$$\texttt{affix}(x) \overset{\text{def}}{=} \bigvee_{m\in\mathcal{M}} m(x) \tag{7}$$

$$\texttt{root}(x) \overset{\text{def}}{=} \bigvee_{r\in\mathcal{R}} r(x) \tag{8}$$

Using these predicates, we can determine whether any two elements are on the same tier:

$$\texttt{same\_tier}(x,y) \overset{\text{def}}{=} (\texttt{p\_slot}(x) \wedge \texttt{p\_slot}(y)) \vee (\texttt{affix}(x) \wedge \texttt{affix}(y)) \vee (\texttt{root}(x) \wedge \texttt{root}(y)) \tag{9}$$

Well-formed tiers are those in which all elements are of the same type:

$$\texttt{WFT} \overset{\text{def}}{=} (\forall x,y) [(x<y) \rightarrow \texttt{same\_tier}(x,y) \wedge (x=y) \rightarrow \texttt{same\_tier}(x,y)] \tag{10}$$

Next, we define well-formed associations between affixes and prosodic templates and between roots and prosodic templates. We begin by defining the consonantal slots in the prosodic template:

$$\texttt{c\_slot}(x) \overset{\text{def}}{=} \texttt{p\_slot}(x) \wedge \neg\texttt{syll}(x) \tag{11}$$

For affixes, we require an association between a C slot and an affixal element, while for roots, we require an association between a consonant slot and a root element.

$$\texttt{WFA}_\texttt{a}(x,y) \overset{\text{def}}{=} \texttt{c\_slot}(y) \wedge \texttt{affix}(x) \wedge x \circ_a y \tag{12}$$

$$\texttt{WFA}_\texttt{r}(x,y) \overset{\text{def}}{=} \texttt{c\_slot}(y) \wedge \texttt{root}(x) \wedge x \circ_r y \tag{13}$$

Well-formed associations require that a single C slot only associate with a single element, either from a root or affix. Notably, the converse isn't required: a single root or affix consonant can occupy multiple C slots on the prosodic template.

$$\texttt{WFA} \overset{\text{def}}{=} (\forall x,y)[x \circ_a y \rightarrow (\texttt{WFA}_\texttt{a}(x,y) \wedge (\nexists z)[z \circ_a y \wedge z \neq x] \wedge (\nexists w)[w \circ_r y]) \wedge x \circ_r y \rightarrow (\texttt{WFA}_\texttt{r}(x,y) \wedge (\nexists z)[z \circ_r y \wedge z \neq x] \wedge (\nexists w)[w \circ_a y])] \tag{14}$$

Finally, I provide an adaptation of the No-Crossing Constraint predicate defined by Jardine (2024):

$$\texttt{NCC} \overset{\text{def}}{=} (\forall x_1,x_2,y_1,y_2)[(x_1 \circ_a x_2 \wedge y_1 \circ_a y_2 \wedge x_1 < x_2) \rightarrow x_2 < y_2 \wedge (x_1 \circ_r x_2 \wedge y_1 \circ_r y_2 \wedge x_1 < x_2) \rightarrow x_2 < y_2] \tag{15}$$

From the above predicates, we define a predicate that determines whether an $\mathfrak{R}$-structure of the form in (1) is a valid autosegmental representation:

$$\texttt{ASR} \overset{\text{def}}{=} \texttt{sep} \wedge \texttt{WFT} \wedge \texttt{WFA} \wedge \texttt{NCC} \tag{16}$$

It is straightforward to see that the $\mathfrak{R}$-structures in Figures 1 and 2 are both valid autosegmental representations according to this predicate.

3.2 Logical Transductions

Having defined our relational structures, we can now define the set of logical transductions which take in unassociated elements of the three tiers and return an associated autosegmental representation. We first define the domain function $\phi_{\texttt{domain}}$. Because we want our transductions to occur for all binyanim, we set $\phi_{\texttt{domain}} \overset{\text{def}}{=} \texttt{True}$. Because the size of the output structure will be the same as the size of the input structure, it suffices to set the domain of the output structure $C \overset{\text{def}}{=} \{1\}$ and $\phi_{\texttt{license}} \overset{\text{def}}{=} \texttt{True}$; this simply indicates that the domain of the output structure is the same as that of the input structure. Next, we define for each relation in $\mathfrak{R}$ the transduction for this relation. Precedence is preserved in our output structure, so we have $\phi_<(x,y) = x < y$. For all relations in $\mathcal{P}$, $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{M}$, these relations hold in the output only if they hold in the input, so for example $\phi_\texttt{m}(x) = m(x)$ for all $m \in \mathcal{M}$; the transduction

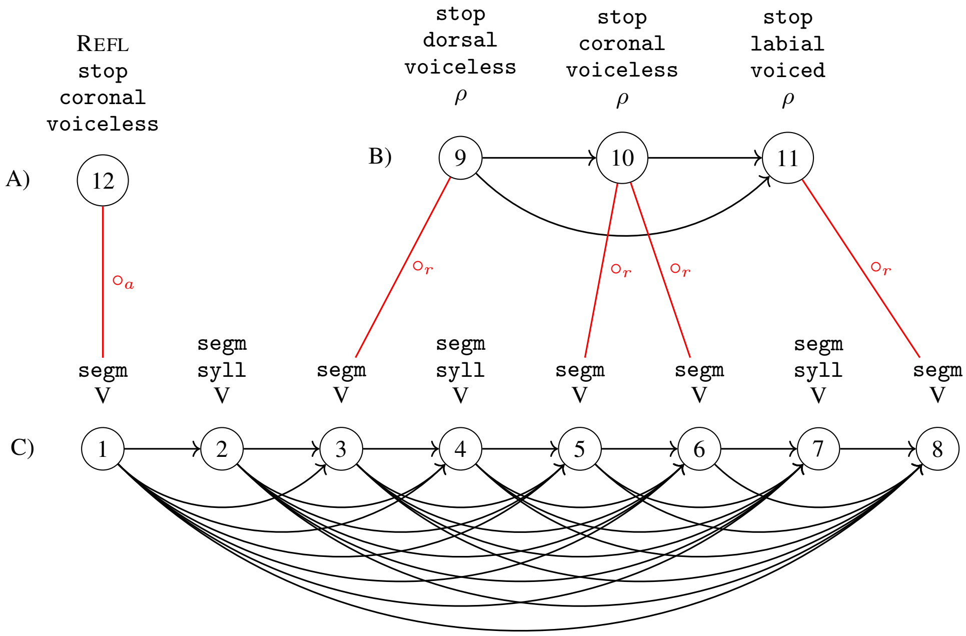


Figure 3: Representations of the affix [t-] (A), root *KTB* (B), CVCVCCVC template (C) and the associations (red) output by our transduction for *taKaTTaB* in binyan V.

formulae are analogous for the other three sets of unary relations. We now turn to the main contribution of the paper: the transductions for affixation $\phi_{\circ_a}$ (§3.2.1) and root association $\phi_{\circ_r}$ (§3.2.2). An example of the transduced associations for *taKaTTaB* in binyan V is given in Figure 3 with the associations highlighted in red; we will return to this example in §3.3.

3.2.1 Affixation: $\phi_{\circ_a}$

For the current purposes, we focus on the association between a single affix and its position in the prosodic template, but it is straightforward to see how this approach would extend to multiple affixes. To determine the appropriate slot(s) in the template, we define a predicate for the ordering relation over consonantal slots using the successor relation:

$$x \triangleleft_c y \overset{\text{def}}{=} \texttt{c\_slot}(x) \wedge \texttt{c\_slot}(y) \wedge x < y \wedge \nexists(z)[x < z < y \wedge \texttt{c\_slot}(z)] \quad (17)$$

We then use this predicate to define the relevant positions of consonants in the prosodic template:

$$\texttt{first\_c}(x) \overset{\text{def}}{=} \texttt{c\_slot}(x) \wedge \nexists(y)[y \triangleleft_c x] \quad (18)$$

$$\texttt{last\_c}(x) \overset{\text{def}}{=} \texttt{c\_slot}(x) \wedge \nexists(y)[x \triangleleft_c y] \quad (19)$$

$$\texttt{second\_c}(x) \overset{\text{def}}{=} \exists(y)[y \triangleleft_c x \wedge \texttt{first\_c}(y)] \quad (20)$$

$$\texttt{third\_c}(x) \overset{\text{def}}{=} \exists(y)[y \triangleleft_c x \wedge \texttt{second\_c}(y)] \quad (21)$$

$$\texttt{fourth\_c}(x) \overset{\text{def}}{=} \exists(y)[y \triangleleft_c x \wedge \texttt{third\_c}(y)] \quad (22)$$

Similarly, we define predicates for the relevant positions in the affix:

$$\texttt{first\_a}(x) \overset{\text{def}}{=} \texttt{affix}(x) \wedge (\nexists z)[\texttt{affix}(z) \wedge z < x] \quad (23)$$

$$\texttt{last\_a}(x) \overset{\text{def}}{=} \texttt{affix}(x) \wedge (\nexists z)[\texttt{affix}(z) \wedge x < z] \quad (24)$$

It is these predicates, which make explicit the position of each of the finitely many consonantal slots and affixal elements, which allow us to define our affixation transduction using only First Order logic. For binyanim IV, V, VI, and VII, the prefix simply fills the first C slot of the prosodic template:

$$\texttt{simple\_prefix}(x, y) \overset{\text{def}}{=} \texttt{first\_c}(y) \wedge (\text{IV}(y) \vee \text{V}(y) \vee \text{VI}(y) \vee \text{VII}(y)) \quad (25)$$

For binyan VIII, I differ from McCarthy in that I associate the affix directly with the second C slot rather than with the first and then flipping it. As discussed in §3.1, however, either approach requires encoding information about the binyan in our representation, and our approach does not require ordered transductions, while a more direct implementation of McCarthy's flop rule would.

$$\texttt{VIII\_infix}(x,y) \overset{\text{def}}{=} \text{VIII}(y) \wedge \texttt{second\_c}(y) \tag{26}$$

Next, the [st-] prefix in binyan X associates the first consonant of the prefix to the first C slot of the prosodic template and the second to the second:

$$\texttt{X\_prefix}(x,y) \overset{\text{def}}{=} \text{X}(y) \wedge ((\texttt{first\_a}(x) \wedge \texttt{first\_c}(y)) \vee (\texttt{last\_a}(x) \wedge \texttt{second\_c}(y))) \tag{27}$$

The infix in binyanim XII and XIV associates with the third C slot in the prosodic template:

$$\texttt{simple\_infix}(x,y) \overset{\text{def}}{=} \texttt{third\_c}(y) \wedge (\text{XII}(y) \vee \text{XIV}(y)) \tag{28}$$

The infix in binyan XIII associates with the third and fourth C slots. Following McCarthy, I assume that this infix consists of a single [w] on the morphological tier, which is associated with two positions in the prosodic template:

$$\texttt{XIII\_infix} \overset{\text{def}}{=} \text{XIII}(y) \wedge (\texttt{third\_c}(y) \vee \texttt{fourth\_c}(y)) \tag{29}$$

Having defined our affix associations for all binyanim which take affixes, we define our overall affixation transduction as the disjunction of these: an association between elements x and y is a valid affix association if x is an element of an affix and one of the predicates defined above is satisfied:

$$\phi_{\circ_a}(x,y) \overset{\text{def}}{=} \texttt{affix}(x) \wedge (\texttt{simple\_prefix}(x,y) \vee \texttt{VIII\_infix}(x,y) \vee \texttt{X\_prefix}(x,y) \vee \texttt{simple\_infix}(x,y) \vee \texttt{XIII\_infix}(x,y)) \tag{30}$$

3.2.2 Root Association: $\phi_{\circ_r}$

We know from McCarthy (1981) that the root consonants will fill the remaining C slots in the prosodic template from left-to-right in general (albeit with binyanim II, V, and XIII violating this). We thus wish to determine whether a given C slot is still available for association after affixation has applied. There are two options for doing so. First, we can follow McCarthy and define it in terms of whether affixation has already applied, as follows:

$$\texttt{unassociated}(x) \overset{\text{def}}{=} \texttt{c\_slot}(x) \wedge \nexists(y)[y \circ_a x] \tag{31}$$

This definition is in line with traditional autosegmental and generative rule-ordering approaches, and could be employed so long as $\phi_{\circ_r}$ is ordered *after* $\phi_{\circ_a}$, so that it can make reference to $\circ_a$. Because we encode information about the binyanim in our representations, however, we can instead define our predicate in terms of the C slots that we know *will* be associated with an affix. This approach allows $\phi_{\circ_r}$ and $\phi_{\circ_a}$ to remain unordered; i.e. they may apply simultaneously. The corresponding predicate is given below.

$$\begin{aligned}\texttt{unassociated}(x) \overset{\text{def}}{=} {} & \texttt{c\_slot}(x) \\ & \wedge ((\neg\, \texttt{first\_c}(x) \wedge \\ & (\text{IV}(x) \vee \text{V}(x) \\ & \vee \text{VI}(x) \vee \text{VII}(x))) \\ & \vee (\text{VIII}(x) \wedge \neg\, \texttt{second\_c}(x)) \\ & \vee (\text{X}(x) \wedge \neg(\texttt{first\_c}(x) \\ & \vee \texttt{second\_c}(x))) \\ & \vee (\neg\, \texttt{third\_c}(x) \\ & \wedge (\text{XII}(x) \vee \text{XIV}(x))) \\ & \vee (\text{XIII}(x) \wedge \neg(\texttt{third\_c}(x) \\ & \vee \texttt{fourth\_c}(x))))\end{aligned} \tag{32}$$

Each portion of the disjunction is straightforwardly derivable from the affixation predicates in §3.2.1; though somewhat redundant, this predicate allows us not to concern ourselves with ordering of our logical transductions. We now define an ordering over unassociated C slots in the prosodic template.

$$\begin{aligned} x \lhd_{\text{UC}} y \overset{\text{def}}{=} {} & \texttt{unassociated}(x) \\ & \wedge \texttt{unassociated}(y) \wedge x < y \\ & \nexists(z)[x < z < y \wedge \texttt{unassociated}(z)]\end{aligned} \tag{33}$$

We use this to define predicates for the positions of the unassociated consonants:

$$\texttt{first\_UC}(x) \stackrel{\text{def}}{=} \texttt{unassociated}(x) \wedge \nexists(y)[y \triangleleft_{\texttt{UC}} x] \tag{34}$$

$$\texttt{last\_UC}(x) \stackrel{\text{def}}{=} \texttt{unassociated}(x) \wedge \nexists(y)[x \triangleleft_{\texttt{UC}} y] \tag{35}$$

$$\texttt{second\_UC}(x) \stackrel{\text{def}}{=} \exists(y)[y \triangleleft_{\texttt{UC}} x \wedge \texttt{first\_c}(y)] \tag{36}$$

$$\texttt{third\_UC}(x) \stackrel{\text{def}}{=} \exists(y)[y \triangleleft_{\texttt{UC}} x \wedge \texttt{second\_c}(y)] \tag{37}$$

$$\texttt{fourth\_UC}(x) \stackrel{\text{def}}{=} \exists(y)[y \triangleleft_{\texttt{UC}} x \wedge \texttt{third\_c}(y)] \tag{38}$$

We similarly define predicates for the position of the root consonants:

$$\texttt{first\_r}(x) \stackrel{\text{def}}{=} \texttt{root}(x) \wedge (\nexists z)[\texttt{root}(z) \wedge z < x] \tag{39}$$

$$\texttt{second\_r}(x) \stackrel{\text{def}}{=} \texttt{root}(x) \wedge (\exists z)[\texttt{root}(z) \wedge z < x] \wedge (\exists z)[\texttt{root}(w) \wedge x < w] \tag{40}$$

$$\texttt{third\_r}(x) \stackrel{\text{def}}{=} \texttt{root}(x) \wedge (\nexists z)[\texttt{root}(z) \wedge x < z] \tag{41}$$

Again, it is these predicates, which make explicit the position of each of the finitely many elements of the prosodic template and consonantal root, which allow us to define the root association transduction using only First Order logic. The first consonant of the root *always* associates with the first available C slot, and the second *always* associates with the second. If there is a fourth unassociated C slot, it will always be associated to the final consonant of the root:

$$\texttt{assoc}(x,y) \stackrel{\text{def}}{=} (\texttt{first\_UC}(y) \wedge \texttt{first\_r}(x)) \vee (\texttt{second\_UC}(y) \wedge \texttt{second\_r}(x)) \vee (\texttt{fourth\_UC}(y) \wedge \texttt{third\_r}(x)) \tag{42}$$

The third consonant of the root will associate with the third unassociated C slot for all binyanim for which McCarthy's erasure rule does not apply, and the second one otherwise:

$$\texttt{erasure}(y) \stackrel{\text{def}}{=} \mathrm{II}(y) \vee \mathrm{V}(y) \vee \mathrm{XII}(y) \tag{43}$$

$$\texttt{third}(x,y) \stackrel{\text{def}}{=} \texttt{third\_UC}(y) \wedge ((\texttt{erasure}(y) \wedge \texttt{second\_r}(x)) \vee (\neg\,\texttt{erasure}(y) \wedge \texttt{third\_r}(x))) \tag{44}$$

Our complete root-pattern association transduction is thus simply given by:

$$\phi_{\circ_r}(x,y) \stackrel{\text{def}}{=} \texttt{assoc}(x,y) \vee \texttt{third}(x,y) \tag{45}$$

3.3 Example

To illustrate the logical transductions defined in the previous section, we consider the derivation of *taKaTTaB* in binyan V. The representations of the prosodic CVCVCCVC template, root *KTB*, and affix [t-] are given in Figure 3.

Following §3.2, the domain of the output structure is the same as the input structure, and all features in $\mathcal{M}, \mathcal{P}, \mathcal{C}$, and $\mathcal{R}$ are preserved, as is the precedence relation $<$; the domain thus remains unchanged. We now turn to our First Order transductions $\phi_{\circ_a}$ and $\phi_{\circ_r}$. We consider the former first for ease of exposition, but recall from §3.2 that these transductions can be — and are in the current analysis — applied simultaneously.

We know that $\texttt{affix}(12) = \texttt{True}$, where 12 refers to domain element 12 (Figure 3 A). We also know that for all elements y in our prosodic template (Figure 3 C), we have $\mathrm{V}(y) = \texttt{True}$. As such, the only predicate in the disjunction of $\phi_{\circ_a}$ that we might satisfy is `simple_prefix`, since all other predicates require a different binyan. From Equation 25, we know that we must also have $\texttt{first\_c}(y)$, which is true of element 1; we thus have $12 \circ_a 1$. No other associations are licensed by $\phi_{\circ_a}$ so this transduction is complete.

Next, we consider our root association relation $\phi_{\circ_r}$. From Equation 32, we have that $\texttt{unassociated}(y) = \texttt{True}$ for $y \in \{3,5,6,8\}$. We begin by considering the general associations given by `assoc` (Equation 42). We have that $\texttt{first\_UC}(3) = \texttt{True}$ and $\texttt{first\_r}(9) = \texttt{True}$, and thus $9 \circ_r$

3. Similarly, `second_UC` $(5) = $ `True` and `second_r` $(10) = $ `True`, so we have $10 \circ_r 5$. Finally, we have `fourth_UC` $(8) = $ `True` and `third_r` $(11) = $ `True`, so $11 \circ_r 8$. This exhausts `assoc`, and we turn to `third`. We have `erasure` $(y) = $ `True` for all y that are domain elements of the template (Figure 3 C), and `third_UC` $(6) = $ `True` and `second_r` $(10) = $ `True`, so by Equation 44, $10 \circ_r 6$, concluding our transduction for $\phi_{\circ_r}$. The output structure is visualized in Figure 3, with the outputs of the logical transductions $\phi_{\circ_r}$ and $\phi_{\circ_a}$ highlighted in red.

4 Discussion

The transductions defined in this paper have served to demonstrate that the constraints of the Arabic system — in particular, the finite length of the consonantal root — make it possible to define transductions corresponding to the autosegmental analysis of McCarthy (1981) using only First Order logic, despite such transductions not being MSO-definable for autosegmental representations of arbitrary length (Jardine, 2017). I highlight two directions for future work.

Firstly, while a majority of roots in Arabic are triconsonantal, the length of the consonantal root varies from 2 to 5, and McCarthy extends his analysis to encompass those with 4. Though this analysis focuses on Modern Standard Arabic, the specifics of the association transductions vary across Arabic dialects, and future work should extend the approach here to encompass this variation. This analysis could also be extended to other Semitic languages, including Hebrew, which allows for even greater complexity in the consonantal root. Importantly, what allows for the present result is the *finite* number of consonants in the root, rather than a *specific* exact number of these consonants. It is thus expected that the association transductions would be straightforwardly First Order definable for roots consisting of some other finite number of consonants, though these transductions may be largely separate from those used for the triconsonantal roots here. Indeed, in extending the present analysis, it will be important to weigh tradeoffs between capturing apparent generalizations and remaining at least MSO-definable.

Secondly, the approach presented here focuses on the logical complexity of autosegmental approaches to root-and-pattern morphology, but a great deal of other approaches exist in the literature (Hudson, 1986; Ussishkin, 2003; Arad, 2005; Tucker, 2011; Alqarni, 2022, i.a.). Several of these approaches involve defining a well-formed output in an Optimality-Theoretic sense rather than defining a *transduction* between a set of unassociated input elements and the associated output. Given the limits of MSO-definability for autosegmental approaches in the general case (Jardine, 2017), and the insights gained by the transductions defined in this paper, it would be interesting to take a parallel approach of logical formalization to these other frameworks. Such an investigation may yield new insight into existing theoretical debates by highlighting which properties of the Arabic system allow for different theories to be computationally tractable: for autosegmental phonology, it is the finite nature of the root, but for Optimility-Theoretic approaches, it may be something else entirely. Considering the different properties on which the tractability of these theories hinges may give new insight into which approaches should be favored from a cognitive perspective.

5 Conclusion

In this paper, I have demonstrated that the logical transductions corresponding to McCarthy's three-tier autosegmental analysis of Arabic derivational morphology are definable with First Order logic, and I introduced the corresponding relational structures and association transductions. This First Order definability is due to the small, finite length of the consonantal root in Arabic, and is in contrast to cases of arbitrary length, where left-to-right autosegmental association is not even MSO-definable (Jardine, 2017). Indeed, the relatively constrained set of possible lengths for Semitic consonantal roots is an interesting property of these languages, and the analysis presented here demonstrates a computational advantage of this property of the system.

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