Supplementary Material for: Tensors over Semirings for Latent-Variable Weighted Logic Programs

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Appendix A - Proofs of Theorems in Main Paper

Lemma 5.1. *For any k, l, $\otimes_{[k;l]}$ distributes over $\oplus$*

Proof. We will proceed by showing that:

$$A \otimes_{[k;l]} (B \oplus C) = (A \otimes_{[k;l]} B) \oplus (A \otimes_{[k;l]} C)$$

Firstly, note that for the left hand side of the equation to be defined, B and C needs to be of matching ranks, and that $B \oplus C$ will be the same rank as both B and C. Therefore, if the left hand side is well defined then both $A \otimes_{[k;l]} B$ and $A \otimes_{[k;l]} C$ is defined and has matching ranks. So the right hand side is defined if and only if the left hand side is defined as well.

$$\begin{aligned}
&[A \otimes_{[j;k]} (B \oplus C)]_{i_1,\dots,i_{k-1},j_1,\dots,j_{l-1},j_{l+1},\dots,j_m,i_{k+1},\dots,i_n} \\
&\quad = \sum_{i_k,j_l} \delta(i_k, j_l) A_{i_1,\dots,i_n} \times (B \oplus C)_{j_1,\dots,j_m} \\
&\quad = \sum_{i_k,j_l} \delta(i_k, j_l) A_{i_1,\dots,i_n} \times (B_{j_1,\dots,j_m} + C_{j_1,\dots,j_m}) \\
&\quad = \sum_{i_k,j_l} \delta(i_k, j_l)(A_{i_1,\dots,i_n} \times B_{j_1,\dots,j_m}) + \delta(i_k, j_l)(A_{i_1,\dots,i_n} \times C_{j_1,\dots,j_m}) \\
&\quad = [(A \otimes_{[k;l]} B) \oplus (A \otimes_{[k;l]} C)]_{i_1,\dots,i_{k-1},j_1,\dots,j_{l-1},j_{l+1},\dots,j_m,i_{k+1},\dots,i_n}
\end{aligned}$$

□

Theorem 5.4. *An item-based description I is correct if*

- *For every grammar G, the mapping $g : \mathcal{D}_{I(G)} \to \mathcal{D}_G$ that maps $d' \in \mathcal{D}_{I(G)}$ to the corresponding $d \in \mathcal{D}_G$ is a bijection with an inverse function f.*
- *For any complete semiring S and weight function w, g and f preserve the values assigned to a derivation:*

$$V^w_G(d) = V^w_{I(G)}(f(d)) \text{ and } V^w_{I(G)}(d') = V^w_G(g(d'))$$

Proof.

$$V^w_{I(G)}(\alpha) = V^w_{I(G)}(goal, \alpha) = \bigoplus_{D \in \text{inner}_\alpha(goal)} V^w_{I(G)}(D) = \bigoplus_{D \in \mathcal{D}_{I(G)}(\alpha)} V^w_G(g(D))$$

Observe that $D \in \mathcal{D}_{I(G)}(\alpha)$ iff $g(D) \in \mathcal{D}_G(\alpha)$ since the rules that appear in the leaves of D, applied from left to right, determines the grammar derivation tree $g(D)$ uniquely via g, and vice versa. Hence,

$$V^w_{I(G)}(\alpha) = \bigoplus_{g(D)\in\mathcal{D}_G(\alpha)} V^w_G(g(D)) = V^w_G(\alpha)$$

□

Theorem 6.1.

$$V(x) = \bigoplus_{\substack{[a_1,\ldots,a_k] \\ s.t. \frac{a_1,\ldots,a_k}{x}}} V(a_1) \otimes [V(a_2), \ldots, V(a_k)]$$

Proof. Recall that by definition, $V(x) = \bigoplus_{D\in inner(x)} V(D)$. For any item derivation D, D is either an axiom or there is some $a_1, \ldots, a_k, b$ s.t. $D \in inner(\frac{a_1 \ldots a_k}{b})$. If D is an axiom, then $inner(D)$ is just a single rule a, and so $V(D) = V(a)$. Else, for each rule $\frac{a_1 \ldots a_k}{x}$

$$\bigoplus_{D\in inner(\frac{a_1\ldots a_k}{x})} V(D) = \bigoplus_{\substack{D_{a_1}\in inner(a_1),\ldots, \\ D_{a_k}\in inner(a_k)}} V(D_{a_1}) \otimes [V(D_{a_2}), \ldots, V(D_{a_k})]$$

$$= \left(\bigoplus_{D_{a_1}\in inner(a_1)} V(D_{a_1})\right) \otimes \left(\bigoplus_{\substack{D_{a_2}\in inner(a_2),\ldots, \\ D_{a_k}\in inner(a_k)}} \bigotimes_{i=2}^{k} V(D_{a_i})\right)$$

$$= \left(\bigoplus_{D_{a_1}\in inner(a_1)} V(D_{a_1})\right) \otimes \left(\bigoplus_{D_{a_2}\in inner(a_2)} V(D_{a_2}), \ldots, \bigoplus_{D_{a_k}\in inner(a_k)} V(D_{a_k})\right)$$

$$= V(a_1) \otimes [V(a_2), \ldots, V(a_k)]$$

Where the last step holds due to the distributive property of the partial semiring.

Since the set $inner(x) = \bigcup_i D_i$ where $D_i \in inner(\frac{a_1 \ldots a_k}{x})$ for all inference rules $\frac{a_1 \ldots a_k}{x}$, we can write the summation over $D \in inner(x)$ as:

$$V(x) = \bigoplus_{D\in inner(x)} V(D)$$

$$= \bigoplus_{\substack{[a_1,\ldots,a_k] \\ \text{s.t.} \frac{a_1,\ldots,a_k}{x}}} \bigoplus_{D\in inner(\frac{a_1\ldots a_k}{x})} V(D)$$

$$= \bigoplus_{\substack{[a_1,\ldots,a_k] \\ \text{s.t.} \frac{a_1,\ldots,a_k}{x}}} V(a_1) \otimes [V(a_2), V(a_3), \ldots, V(a_k)]$$

Where the last line is obtained by replacing the inner part of the expression with the equality obtained from the previous part of the proof. □

Lemma 6.2. *Let V and Z be defined on a commutative semiring $\mathbb{S}$ and let $O \in outer_\alpha(x)$ and $T \in inner_\alpha(x)$. If combining O and T in the obvious way results in the complete derivation D then*

$$V(D) = V(T) \otimes^* Z(O)$$

Proof. To simplify notation of the indices, let $\mathbf{i}$ stand for a list of indices $i_1, \ldots, i_n$ for some n. We will also use $\mathbf{d}^i$ to denote a list $d_1^i, \ldots d_{n_i}^i$ and $\mathbf{d}$ to denote $\mathbf{d}^1, \ldots, \mathbf{d}^n$. $\delta(\mathbf{i}, \mathbf{j}) = \prod_{k=1}^{n} \delta(i_k, j_k)$.

We will proceed by induction on the parse tree. Base case is where $x = goal$, $T = D$ and O is empty. Then $V(T) = V(D)$ and $Z(O) = I_S$. $V(D) \otimes^* I_S = V(D)$ by the definition of I_S which proves the statement.

Otherwise T has a parent tree $T_p = \langle y : T_1, \ldots, T_n \rangle$ where $T = T_k$. Furthermore, $T_p \in \text{inner}_\alpha(y)$, $O_p \in \text{outer}_\alpha(y)$ and by induction hypothesis $V(D) = V(T_p) \otimes^* Z(O_p)$.

Since $T_p \in \text{inner}_\alpha(y)$ we know that

$$V(T_p) = V(T_1) \otimes [V(T_2), \ldots, V(T_m)]$$

So

$$V(D) = (V(T_1) \otimes [V(T_2), \ldots, V(T_m)]) \otimes^* Z(O_p)$$

The proof progresses by calculating the value for $[V(D)]_i$ based on the above term and shows that this is equal to the value of $[V(T) \otimes^* Z(O)]_i$.

Let:

$$V(T_1) \in \mathbb{S}^{\mathbf{e},\mathbf{f}} \qquad V(T_i) \in \mathbb{S}^{e_i,\mathbf{d}^i}$$
$$Z(O_p) \in \mathbb{S}^{\mathbf{d},\mathbf{f},s} \qquad V(D) \in \mathbb{S}^s$$

Then:

$$\begin{aligned} V[(T_p)]_{\mathbf{d},\mathbf{f}} &= [V(T_1) \otimes (V(T_2), \ldots, V(T_m))]_{\mathbf{d},\mathbf{f}} \\ &= \sum_{\mathbf{e},\mathbf{e}'} V(T_1)_{\mathbf{e},\mathbf{f}} \times \prod_{i=2}^{m} \delta(e_i, e_i') V(T_i)_{e_i',\mathbf{d}^i} \end{aligned}$$

$$\begin{aligned} [V(D)]_s = [V(T_p) \otimes^* Z(O_p)]_s = \\ & \sum_{\mathbf{e},\mathbf{e}',\mathbf{d},\mathbf{d}'\mathbf{f},\mathbf{f}'} V(T_1)_{\mathbf{e},\mathbf{f}} \times \left(\prod_{i=2}^{m} \delta(e_i, e_i') V(T_i)_{e_i,\mathbf{d}^i} \right) \\ & \times \delta(\mathbf{d}, \mathbf{d}') \delta(\mathbf{f}, \mathbf{f}') Z(O_p)_{\mathbf{d},\mathbf{f},s} \end{aligned}$$

Now we will proceed to prove that this term is equal to $V(T_k) \otimes^* Z(O)$. Let $I_{T_k} \in \mathbb{S}^{e_k',\mathbf{d}^k,s,e_k,\mathbf{d}^k,s}$. We will calculate the value of the outside term in sections. Let $A = V(T_1) \otimes_k$

$(I_{T_k}, V(T_{k+1}), \ldots, V(T_n))$. Then,

$$A_{e_1,\ldots,e_{k-1},\mathbf{d}^k,s,\hat{e}_k,\hat{\mathbf{d}}_k,\hat{s},\mathbf{d}^{k+1},\ldots,\mathbf{d}^n,\mathbf{f}} =$$
$$A^\pi_{e_1,\ldots,e_{k-1},\mathbf{d}^k,\mathbf{d}^{k+1},\ldots,\mathbf{d}^n,\mathbf{f},s,\hat{e}_k,\hat{\mathbf{d}}_k,\hat{s}} =$$
$$\sum_{\substack{e_k,\ldots,e_n\\ e'_k,\ldots,e'_n}} V(T_1)_{\mathbf{e},\mathbf{f}} \times \delta(e_k, e'_k)\delta(\mathbf{d}^k, \hat{\mathbf{d}}^k)\delta(s, \hat{s}) \times \prod_{i=k+1}^{m} \delta(e_i, e'_i)V(T_i)_{e'_i,\mathbf{d}^i}$$
$$[A^\pi \otimes (V(T_2), \ldots, V(T_{k-1}))]_{\mathbf{d},\mathbf{f},s,\hat{e}_k,\hat{\mathbf{d}}^k,\hat{s}} =$$
$$\sum_{\mathbf{e},\mathbf{e}'} V(T_1)_{\mathbf{e},\mathbf{f}} \times \prod_{\substack{i=2\\ i\neq k}}^{n} V(T_i)_{e'_i,\mathbf{d}^i} \times$$
$$\delta(\mathbf{e}, \mathbf{e}') \times \delta(e_k, \hat{e}_k) \times \delta(\mathbf{d}^k, \hat{\mathbf{d}}^k) \times \delta(s, \hat{s})$$
$$[Z(O)]_{\hat{e}_k,\hat{\mathbf{d}}^k,\hat{s}} = \sum_{\substack{\mathbf{e},\mathbf{e}',\mathbf{d},\mathbf{d}'\\ \mathbf{f},\mathbf{f}',s,s'}} V(T_1)_{\mathbf{e},\mathbf{f}} \times \prod_{\substack{i=2\\ i\neq k}}^{n} V(T_i)_{e'_i,\mathbf{d}^i} \times Z(O_p)_{\mathbf{d}',\mathbf{f}',s'}$$
$$\times\, \delta(\mathbf{e}, \mathbf{e}') \times \delta(e_k, \hat{e}_k) \times \delta(\mathbf{d}^k, \hat{\mathbf{d}}^k) \times \delta(s, \hat{s})$$
$$\times\, \delta(\mathbf{d}, \mathbf{d}') \times \delta(\mathbf{f}, \mathbf{f}') \times \delta(s, s')$$
$$[V(T_k) \otimes^* Z(O)]_{\hat{s}} =$$
$$\sum_{\substack{\mathbf{e},\mathbf{e}',\mathbf{d},\mathbf{d}'\\ \mathbf{f},\mathbf{f}',s,s'\\ \hat{e}_k,\hat{\mathbf{d}}^k,e''_k,\mathbf{d}^{k''}}} V(T_k)_{e''_k,\mathbf{d}^{k''}} \times V(T_1)_{\mathbf{e},\mathbf{f}} \times \prod_{\substack{i=2\\ i\neq k}}^{n} V(T_i)_{e'_i,\mathbf{d}^i} \times Z(O_p)_{\mathbf{d}',\mathbf{f}',s'}$$
$$\times\, \delta(\mathbf{e}, \mathbf{e}') \times \delta(e_k, \hat{e}_k) \times \delta(\mathbf{d}^k, \hat{\mathbf{d}}^k) \times \delta(s, \hat{s})$$
$$\times\, \delta(\mathbf{d}, \mathbf{d}') \times \delta(\mathbf{f}, \mathbf{f}') \times \delta(s, s') \times \delta(e''_k, \hat{e}_k) \times \delta(\mathbf{d}^{k''}, \hat{\mathbf{d}}^k)$$
$$= \sum_{\mathbf{e},\mathbf{e}',\mathbf{d},\mathbf{d}'\mathbf{f},\mathbf{f}'} V(T_1)_{\mathbf{e},\mathbf{f}} \times \prod_{i=2}^{m} V(T_i)_{e_i,\mathbf{d}^i} \times Z(O_p)_{\mathbf{d},\mathbf{f},\hat{s}}$$
$$\times\, \delta(\mathbf{e}, \mathbf{e}') \times \delta(\mathbf{d}, \mathbf{d}') \times \delta(\mathbf{f}, \mathbf{f}')$$

Which completes the proof. The last simplification step is obtained by replacing $\hat{e}_k$ and e''_k with e_k, $\hat{\mathbf{d}}^k$ and $\mathbf{d}^{k''}$ with $\mathbf{d}^k$ and s and s' with $\hat{s}$ since these need to be equal for any term to contribute to the final sum. The commutativity of $\mathbb{S}$ then allows $V(T_k)_{e_k,\mathbf{d}^k}$ to be moved to its place in the sequence.

□

Theorem 6.4. *If x is the goal item, then $Z(x) = I_s$. Else,*

$$Z(x) = \bigoplus_{\substack{j,a_1,..,a_k,b \text{ s.t.}\\ \frac{a_1\ldots a_k}{b} \text{ and } x=a_j}} (V(a_1) \otimes_k [I_{a_k}, V(a_{k+1}), \ldots, V(a_n)])^\pi$$
$$\otimes (V(a_2), \ldots, V(a_{k-1})) \otimes^* Z(b)$$

Proof. by definition $Z(x) = \bigoplus_{D\in outer(x)} Z(D)$. Either x is a goal item, in which case $Z(x) = Z() = I_S$.

Otherwise the outer trees $outer(x)$ could be written as the union of outer trees $outer\left(k, \frac{a_1\ldots a_n}{b}\right)$ for each rule $\frac{a_1\ldots a_n}{b}$ where $a_k = x$ for some k. Hence:

$$Z(x) = \bigoplus_{\substack{j,a_1,..,a_k,b \text{ s.t.}\\ \frac{a_1\ldots a_k}{b} \text{ and } x=a_j}} \bigoplus_{D\in outer\left(k,\frac{a_1\ldots a_n}{b}\right)} Z(D)$$

For the inner part of this equation we have:

$$\bigoplus_{D \in outer\left(k, \frac{a_1 \dots a_n}{b}\right)} Z(D) =$$

$$\bigoplus_{D_b \in outer(b)} \bigoplus_{\substack{D_{a_1} \in inner(a_1), \dots, \\ D_{a_{k-1}} \in inner(a_{k-1})}} \bigoplus_{\substack{D_{a_{k+1}} \in inner(a_{k+1}), \dots, \\ D_{a_n} \in inner(a_n)}}$$

$$\Big(V(D_{a_1}) \otimes_k \Big[I_{D_{a_k} \times d_S}, V(D_{a_{k+1}}), \dots, V(D_{a_n})\Big]\Big)^{\pi}$$

$$\otimes \big(V(D_{a_2}), \dots, V(D_{a_{k-1}})\big) \otimes^* Z(D_b)$$

Since $\oplus$ distributes over $\otimes$, this can rewritten as

$$\bigoplus_{D \in outer\left(k, \frac{a_1 \dots a_n}{b}\right)} Z(D) =$$

$$\left(\bigoplus_{\substack{D_{a_1} \in \\ inner(a_1)}} V(D_{a_1}) \otimes_k \left[I_{D_{a_k}}, \bigoplus_{\substack{D_{a_{k+1}} \in \\ inner(a_{k+1})}} V\left(D_{a_{k+1}}\right), \dots, \bigoplus_{\substack{D_{a_n} \in \\ inner(a_n)}} V\left(D_{a_n}\right) \right] \right)^{\pi}$$

$$\otimes \left(\bigoplus_{D_{a_2} \in inner(a_2)} V(D_{a_2}), \dots, \bigoplus_{D_{a_{k-1}} \in inner(a_{k-1})} V(D_{a_{k-1}}) \right)$$

$$\otimes^* \bigoplus_{D_b \in outer(b)} Z(D_b)$$

And since $V(a_i)$ and $Z(D_b)$ are defined as the summation of their inner and outer trees respectively

$$\bigoplus_{D \in outer\left(k, \frac{a_1 \dots a_n}{b}\right)} Z(D) =$$

$$(V(a_1) \otimes_k [I_{a_k}, V(a_{k+1}), \dots, V(a_n)])^{\pi} \otimes (V(a_2), \dots, V(a_{k-1})) \otimes^* Z(b)$$

Replacing the inner part of the previous equation with this term gives us the desired equality, completing the proof. □

Appendix B - Inside and outside calculations for looping buckets

In computing the inside and outside values with an item-based description, we assume a pre-computed ordering over items in the form of *buckets*. For items x and y, we write $bucket(x) \leq bucket(y)$ if the value of y depends on the value of x. So far we have assumed that items could be simply sorted so that no item directly or indirectly depends on itself, and given the inside and outside formulas accordingly. In this section we give the equivalent formulas for items in *looping buckets*. Items in a looping bucket depend on each other and computing their values might require an infinite sum. Our presentation and proofs both follow that of Goodman (1998).

For an item x in a looping bucket B, let the *generation* of a derivation tree x to be the maximum number of items in B that could appear in a single path from the root to a leaf. This intuitively provides an ordering for processing a potentially infinite number of trees by starting from generation 0 and incrementally adding larger and larger trees. We will denote the set of inner trees of x with generation at most g with $inner_{\leq}(x, B)$ Adding up the values of all inner trees of x that have generation at most g then gives us an approximation for the true inner value of x, and the approximation gets better as g gets larger. Formally, we define a *g generation value* for an item x in bucket B as:

$$V_{\leq g}(x,B) = \bigoplus_{D \in inner_{\leq g}(x,B)} V(D)$$

For ω-continuous semirings, the infinite sum is equal to the supremum of the partial sums (Kuich 1997, 613), hence (Goodman 1999, 589):

$$V(x) = \bigoplus_{D \in inner(x)} V(D) = \sup_g V_{\leq g}(x,B)$$

Fortunately, tensors of semirings of set dimensions are ω-continuous as long as the underlying semiring is ω-continuous. We give the necessary definitions to establish this property:

Definition 1. *(Kuich 1997, 611) A semiring is* ***naturally ordered*** *if there is a partial ordering $\sqsubseteq$ such that $x \sqsubseteq y$ iff there is a z s.t. $x \oplus z = y$.*

Definition 2. *(Kuich 1997, 612) A naturally ordered complete semiring is ω-continuous if for any sequence $x_1, x_2, \ldots$ and for any constant y, if for all n, $\bigoplus_{0 \leq i \leq n} x_i \sqsubseteq y$ then $\bigoplus_i x_i \sqsubseteq y$*

Notice that for the set of tensors in $\mathbb{S}^{\mathbf{d}}$ where $\mathbf{d}$ is an arbitrary list of positive integers, if the underlying semiring has a natural ordering then this could be extended straightforwardly to $\mathbb{S}^{\mathbf{d}}$ by the following rule: $\mathbf{X} \sqsubseteq \mathbf{Y}$ iff $\mathbf{X_i} \sqsubseteq \mathbf{Y_i}$ for all indices $\mathbf{i}$. It is straightforward to check that if the underlying semiring is ω-continuous, then $\mathbb{S}^{\mathbf{d}}$ is ω-continuous as well.

Goodman (1999) gives a formula for $V_{\leq g}(x,B)$ in order to compute or approximate the supremum. Below we give the analogous formula for partial semirings:

Theorem B.1. *For items x in a looping bucket B and the generation $g \geq 1$*

$$V_{\leq g}(x,B) = \bigoplus_{\substack{[a_1,\ldots,a_k] \\ s.t.\ \frac{a_1,\ldots,a_k}{x}}} K_g(a_1,B) \otimes [K_g(a_2,B), \ldots, K_g(a_k,B)]$$

Where

$$K_g(a,B) = \begin{cases} V(a) & \text{if } a \notin B \\ V_{\leq g-1}(a,B) & \text{if } a \in B \end{cases}$$

Proof.

$$\begin{aligned}
V_{\leq g}(x,B) &= \bigoplus_{D \in inner_{\leq g}(x,B)} V(D) \\
&= \bigoplus_{\substack{[a_1,\ldots,a_k] \\ \text{s.t.}\ \frac{a_1,\ldots,a_k}{x}}} \bigoplus_{\substack{D_{a_1} \in inner_{\leq g-1}(a_1,B),\ldots, \\ D_{a_k} \in inner_{\leq g-1}(a_k,B)}} V\left(\langle x : D_{a_1}, \ldots D_{a_k} \rangle\right) \\
&= \bigoplus_{\substack{[a_1,\ldots,a_k] \\ \text{s.t.}\ \frac{a_1,\ldots,a_k}{x}}} \bigoplus_{\substack{D_{a_1} \in inner_{\leq g-1}(a_1,B),\ldots, \\ D_{a_k} \in inner_{\leq g-1}(a_k,B)}} V(D_{a_1}) \otimes [V(D_{a_2}), \ldots, V(D_{a_k})] \\
&= \bigoplus_{\substack{[a_1,\ldots,a_k] \\ \text{s.t.}\ \frac{a_1,\ldots,a_k}{x}}} \bigoplus_{D_{a_1} \in inner_{\leq g-1}(a_1,B)} V(D_{a_1}) \\
&\qquad \otimes \left[\bigoplus_{D_{a_2} \in inner_{\leq g-1}(a_2,B)} V(D_{a_2}), \ldots, \bigoplus_{D_{a_k} \in inner_{\leq g-1}(a_k,B)} V(D_{a_k}) \right] \\
&= \bigoplus_{\substack{[a_1,\ldots,a_k] \\ \text{s.t.}\ \frac{a_1,\ldots,a_k}{x}}} V_{\leq g-1}(a_1,B) \otimes [V_{\leq g-1}(a_2,B), \ldots, V_{\leq g-1}(a_k,B)]
\end{aligned}$$

Note that if a_i is not in the bucket B then $V_{\leq g-1}(a_i, B) = V(a_i)$, hence $V_{\leq g-1}(a_i, B)$ can be replaced with $K_g(a_i, B)$, completing the proof. □

We will follow a similar strategy for computing the outside values of items that belong to a looping bucket. The only difference is the slight difference in the definition of the generation of of the tree. If $D \in \mathit{outer}(x)$ where x belongs to a looping bucket B, then the generation of D is maximum number of items that could appear in a single path from the root to x, where x is included in the count. Let

$$Z_{\leq g}(x, B) = \bigoplus_{D \in \mathit{outer}_{\leq g}(x,B)} Z(D)$$

Theorem B.2. *For items x in a looping bucket B and the generation $g \geq 1$*

$$Z_{\leq g}(x, B) = \bigoplus_{\substack{j,a_1,..,a_k,b \text{ s.t.} \\ \frac{a_1 \cdots a_k}{b} \text{ and } x=a_j}} (V(a_1) \otimes_k [I_{a_k}, V(a_{k+1}), \ldots, V(a_n)])^\pi \otimes [(V(a_1), \ldots, V(a_{k-1})] \otimes^* H_g(b, B)$$

Where π is defined as in Theorem 6.4 and

$$H_g(b, B) = \begin{cases} Z(b) & \text{if } b \notin B \\ Z_{\leq g-1}(b, B) & \text{if } b \in B \end{cases}$$

Proof.

$$\begin{aligned}
Z_{\leq g}(x, B) &= \bigoplus_{D \in \mathit{outer}_{\leq g}(x,B)} Z(D) \\
&= \bigoplus_{\substack{j,a_1,..,a_k,b \text{ s.t.} \\ \frac{a_1 \cdots a_k}{b} \text{ and } x=a_j}} \bigoplus_{D \in \mathit{outer}_{\leq g-1}\left(k, \frac{a_1 \ldots a_n}{b}\right)} Z(D) \\
&= \bigoplus_{\substack{j,a_1,..,a_k,b \text{ s.t.} \\ \frac{a_1 \cdots a_k}{b} \text{ and } x=a_j}} \bigoplus_{D_b \in \mathit{outer}_{\leq g-1}(b)} \bigoplus_{\substack{D_{a_1} \in \mathit{inner}(a_1), \ldots, \\ D_{a_{k-1}} \in \mathit{inner}(a_{k-1})}} \bigoplus_{\substack{D_{a_{k+1}} \in \mathit{inner}(a_{k+1}), \ldots, \\ D_{a_n} \in \mathit{inner}(a_n)}} \\
&\qquad \left(V(D_{a_1}) \otimes_k \left[I_{D_{a_k} \times d_S}, V(D_{a_{k+1}}), \ldots, V(D_{a_n})\right]\right)^\pi \\
&\qquad \otimes \left(V(D_{a_2}), \ldots, V(D_{a_{k-1}})\right) \otimes^* Z_{\leq g}(D_b, B) \\
&= \bigoplus_{\substack{j,a_1,..,a_k,b \text{ s.t.} \\ \frac{a_1 \cdots a_k}{b} \text{ and } x=a_j}} \left(\bigoplus_{\substack{D_{a_1} \in \\ \mathit{inner}(a_1)}} V(D_{a_1}) \otimes_k \left[I_{D_{a_k}}, \bigoplus_{\substack{D_{a_{k+1}} \in \\ \mathit{inner}(a_{k+1})}} V(D_{a_{k+1}}), \ldots, \bigoplus_{\substack{D_{a_n} \in \\ \mathit{inner}(a_n)}} V(D_{a_n}) \right] \right)^\pi \\
&\qquad \otimes \left(\bigoplus_{D_{a_2} \in \mathit{inner}(a_2)} V(D_{a_2}), \ldots, \bigoplus_{D_{a_{k-1}} \in \mathit{inner}(a_{k-1})} V(D_{a_{k-1}}) \right) \\
&\qquad \otimes^* \bigoplus_{D_b \in \mathit{outer}_{\leq g-1}(b)} Z_{\leq g-1}(D_b, B) \\
&= \bigoplus_{\substack{j,a_1,..,a_k,b \text{ s.t.} \\ \frac{a_1 \cdots a_k}{b} \text{ and } x=a_j}} (V(a_1) \otimes_k [I_{a_k}, V(a_{k+1}), \ldots, V(a_n)])^\pi \\
&\qquad \otimes [(V(a_2), \ldots, V(a_{k-1})] \otimes^* Z_{\leq g-1}(b, B)
\end{aligned}$$

Like the inner case, note that for an item b not in the looping bucket b, $Z_{\leq g-1}(b, B) = Z(b)$, hence we can replace $Z_{\leq g-1}(b, B)$ with $H_g(b, B)$, completing the proof. □

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