Categorial grammar, modalities and algebraic semantics

Koen Versmissen
Onderzoeksinstituut voor Taal en Spraak
Universiteit Utrecht
Trans 10
3512 JK Utrecht
Netherlands
koen.versmissen@let.ruu.nl

Abstract

This paper contributes to the theory of substructural logics that are of interest to categorial grammarians. Combining semantic ideas of Hepple [1990] and Morrill [1990], proof-theoretic ideas of Venema [1993b; 1993a] and the theory of equational specifications, a class of *resource-preserving* logics is defined, for which decidability and completeness theorems are established.

1 Introduction

The last decade has seen a keen revival of investigations into the suitability of using categorial grammars as theories of natural language syntax and semantics. Initially, this research was for the larger part confined to the classical categorial calculi of Ajdukiewicz [1935] and Bar-Hillel [1953], and, in particular, the Lambek calculus **L** [Lambek, 1958], [Moortgat, 1988] and some of its close relatives.

Although it turned out to be easily applicable to fairly large sets of linguistic data, one couldn't realistically expect the Lambek calculus to be able to account for all aspects of grammar. The reason for this is the diversity of the constructions found in natural language. The Lambek calculus is good at reflecting surface phrase structure, but runs into problems when other linguistic phenomena are to be described. Consequently, recent work in categorial grammar has shown a trend towards diversification of the ways in which the linguistic algebra is structured, with an accompanying ramification of proof theory.

One of the main innovations of the past few years has been the introduction of unary type connectives, usually termed *modalities*, that are used to reflect certain special features linguistic entities may possess. This strand of research originates with Morrill [1990], who adds to **L** a unary connective $\Box$ with the following proof rules:

$$\frac{\Gamma, \mathsf{B}, \Gamma' \vdash \mathsf{A}}{\Gamma, \Box\mathsf{B}, \Gamma' \vdash \mathsf{A}}\,[\Box L] \qquad \frac{\Box\Gamma \vdash \mathsf{A}}{\Box\Gamma \vdash \Box\mathsf{A}}\,[\Box R]$$

$\Box\Gamma$ here denotes a sequence of types all of which have $\Box$ as their main connective. The **S4**-like modality $\Box$ is introduced with the aim of providing an appropriate means of dealing with certain intensional phenomena. Consequently, $\Box$ inherits Kripke's possible world semantics for modal logic. The proof system which arises from adding Morrill's left and right rules for $\Box$ to the Lambek calculus **L** will be called **L**$\Box$.

Hepple [1990] presents a detailed investigation into the possibilities of using the calculus **L**$\Box$ to account for purely syntactic phenomena, notably the well-known *Island Constraints* of Ross [1967]. Starting from the usual interpretation of the Lambek calculus in semigroups L, where types are taken to denote subsets of L, he proposes to let $\Box$ refer to a fixed subsemigroup $L_\Box$ of L, which leads to the following definition of its semantics:

$$[\![\Box\mathsf{A}]\!] = [\![\mathsf{A}]\!] \cap L_\Box$$

As we have shown elsewhere [Versmissen, 1992][1] the calculus **L**$\Box$ is sound with respect to this semantics, but not complete. This can be remedied by

[1] This paper discusses semigroup semantics for **L** and **L**$\Box$ in detail, and is well-suited as an easy-going introduction to the ideas presented here. It is available by anonymous ftp from `ftp.let.ruu.nl` in directory `/pub/ots/papers/versmissen`, files `adding.dvi.Z` and `adding.ps.Z`.

replacing the rule [$\Box$R] with the following stronger version:

$$\frac{\Gamma_1 \vdash \Box B_1 \quad \dots \quad \Gamma_n \vdash \Box B_n \quad \Gamma_1, \dots, \Gamma_n \vdash A}{\Gamma_1, \dots, \Gamma_n \vdash \Box A} [\Box R']$$

Hepple [1990] also investigates the benefits of using the so-called *structural modalities* originally proposed in [Morrill *et al.*, 1990], for the description of certain discontinuity and dislocality phenomena. The idea here is that such modalities allow a limited access to certain structural rules. Thus, we could for example have a permutation modality $\Box_P$ with the following proof rule (in addition to [$\Box_P$L] and [$\Box_P$R′] as before):

$$\frac{\Gamma[\Box_P A, B] \vdash C}{\Gamma[B, \Box_P A] \vdash C} \updownarrow$$

The symbol $\updownarrow$ here indicates that the inference is valid in both directions. The interpretation of $\Box_P$ would then be taken care of by a subsemigroup $L_{\Box_P}$ of L having the property that $x \cdot y = y \cdot x$ whenever $x \in L_{\Box_P}$ or $y \in L_{\Box_P}$.

Alternatively, one could require *all* types in such an inference to be boxed:

$$\frac{\Gamma[\Box_P A, \Box_P B] \vdash C}{\Gamma[\Box_P B, \Box_P A] \vdash C} \updownarrow$$

In this case, $L_{\Box_P}$ would have to be such that $x \cdot y = y \cdot x$ whenever $x, y \in L_{\Box_P}$.

Closely related to the use of structural modalities is the trend of considering different kinds of product connectives, sometimes combined into a single system. For example, Moortgat & Morrill [1992] present an account of dependency structure in terms of headed prosodic trees, using a calculus that possesses two product operators instead of just one. On the basis of this, Moortgat [1992] sketches a landscape of substructural logics parametrized by properties such as commutativity, associativity and dependency. He then goes on to show how structural modalities can be used to locally enhance or constrain the possibilities of type combination. Morrill [1992] has a non-associative prosodic calculus, and uses a structural modality to reintroduce associativity at certain points.

The picture that emerges is the following. Instead of the single product operator of **L**, one considers a range of different product operators, reflecting different modes of linguistic structuring. This results in a landscape of substructural logics, which are ultimately to be combined into a single system. Specific linguistic phenomena are given an account in terms of type constructors that are specially tailored for their description. On certain occasions it is necessary for entities to 'escape' the rules of the type constructor that governs their behaviour. This is achieved by means of structural modalities, which license controlled travel through the substructural landscape.

Venema [1993a] proves a completeness theorem, with respect to the mentioned algebraic interpretation, for the Lambek calculus extended with a permutation modality. He modifies the proof system by introducing a type constant Q which refers explicitily to the subalgebra $\mathbf{L}_\Box$. This proof system is adapted to cover a whole range of substructural logics in [Venema, 1993b]. However, the semantics given in that paper, which is adopted from Došen [1988; 1989], differs in several respects from the one discussed above. Most importantly, models are required to possess a partial order with a well-behaved interaction with the product operation. In the remainder of this paper we will give a fairly general definition of the notion of a *resource-preserving* logic. The proof theory of these logics is based on that of Venema, while their semantics, with respect to which a completeness theorem will be established, is similar to that of Hepple and Morrill.

2 Resource-preserving logics with structural modalities

2.1 Syntax

The languages of the logics that will be considered here are specified by the following parameters:

- ▷ Three finite, disjoint index sets $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$;
- ▷ A finite set **B** of *basic types.*

Given these, we define the following sets of expressions:

- ▷ The set of binary type connectives $\mathbf{C} = \{/_i, \backslash_i\}_{i \in \mathcal{I}}$;
- ▷ Two sets of unary type connectives $\mathbf{M}_\triangle = \{\triangle_j\}_{j \in \mathcal{J}}$ and $\mathbf{M}_\nabla = \{\nabla_k\}_{k \in \mathcal{K}}$;
- ▷ The set of type constants $\mathbf{Q} = \{Q_j\}_{j \in \mathcal{J}} \cup \{Q_k\}_{k \in \mathcal{K}}$;
- ▷ The set of types **T**, being the inductive closure of $\mathbf{B} \cup \mathbf{Q}$ under $\mathbf{C} \cup \mathbf{M}_\triangle \cup \mathbf{M}_\nabla$;
- ▷ The set of structural connectives $\mathbf{SC} = \{\circ_i\}_{i \in \mathcal{I}}$;
- ▷ The set of *structures* **S**, being the inductive closure of **T** under **SC**;
- ▷ The set of sequents $\{\Gamma \vdash A \mid \Gamma \in \mathbf{S}, A \in \mathbf{T}\}$.

The division of the unary type connectives into two sets $\mathbf{M}_\triangle$ and $\mathbf{M}_\nabla$ reflects the alternatives mentioned in Section 1. Modalities $\triangle_j$ are those whose structural rules only apply when all types involved are prefixed with them, whereas only a single type prefixed with ∇_k needs to be involved in order for the accompanying structural rules to be applicable.

2.2 Equational specifications

We will use equational specifications to describe the structural behaviour of connectives and modalities, as well as the algebraic structures in which these are interpreted. To start with, we recall several important definitions and results.

A *signature* Σ is a collection of function symbols, each of which has a fixed arity. Let $\mathcal{V}$ be a countably infinite set of variables. The *term algebra* $\mathcal{T}(\Sigma,\mathcal{V})$ is defined as the inductive closure of $\mathcal{V}$ under Σ. An *equational specification* is a pair $(\Sigma,\mathcal{E})$ where Σ is a signature and $\mathcal{E}$ is a set of equations $s = t$ of terms $s,t \in \mathcal{T}(\Sigma,\mathcal{V})$. A Σ-*algebra* $\mathcal{A}$ is a set A together with functions $F^{\mathcal{A}} : A^n \to A$ for all n-ary function symbols $F \in \Sigma$. A Σ-algebra $\mathcal{A}$ is a *model* for a set of equations $\mathcal{E}$ over $\mathcal{T}(\Sigma,\mathcal{V})$, written as $\mathcal{A} \models \mathcal{E}$, if every equation of $\mathcal{E}$ holds in $\mathcal{A}$. A $(\Sigma,\mathcal{E})$-algebra is a Σ-algebra that is a model for $\mathcal{E}$.

Let $\mathcal{E}$ be an equational specification. Then we define $\mathcal{E}_{\Delta_j}$ to be the equational specification obtained from $\mathcal{E}$ by prefixing each variable occurrence with Δ_j. The equational specification $\mathcal{E}_{\nabla_k}$ is defined as follows (where $\mathcal{V}(F=G)$ denotes the set of variables occurring in $F=G$):

$$\begin{aligned}(F{=}G)[x{\leftarrow}\nabla_k x] &\equiv_D F[x{\leftarrow}\nabla_k x]{=}G[x{\leftarrow}\nabla_k x]\\ (F{=}G)_{\nabla_k} &\equiv_D \textstyle\bigcup_{x\in\mathcal{V}(F=G)}(F{=}G)[x{\leftarrow}\nabla_k x]\\ \mathcal{E}_{\nabla_k} &\equiv_D \textstyle\bigcup_{E\in\mathcal{E}} E_{\nabla_k}\end{aligned}$$

To give a concrete example of these definitions, let $\mathcal{E}$ consist of the following two equations:

$$\begin{aligned}x+y &= y+x\\ x+(y+z) &= (x+y)+z\end{aligned}$$

Then $\mathcal{E}_{\Delta_j}$ contains these two:

$$\begin{aligned}\Delta_j x+\Delta_j y &= \Delta_j y+\Delta_j x\\ \Delta_j x+(\Delta_j y+\Delta_j z) &= (\Delta_j x+\Delta_j y)+\Delta_j z\end{aligned}$$

whereas $\mathcal{E}_{\nabla_k}$ is comprised of five equations in all:

$$\begin{aligned}\nabla_k x+y &= y+\nabla_k x\\ x+\nabla_k y &= \nabla_k y+x\\ \nabla_k x+(y+z) &= (\nabla_k x+y)+z\\ x+(\nabla_k y+z) &= (x+\nabla_k y)+z\\ x+(y+\nabla_k z) &= (x+y)+\nabla_k z\end{aligned}$$

We will call a term equation *resource-preserving* if each variable occurs the same number of times on both sides of the equality sign. An equational specification is resource-preserving if all of its member equations are. Note that this definition encompasses the important cases of commutativity and associativity. On the other hand, well-known rules such as weakening and contraction can't be modelled by resource-preserving equations. Not only do they fail to be resource-preserving in the strict sense introduced here, but also they are one-way rules that would have to be described by means of rewrite rules rather than equations.

2.3 Resource-preserving logics

A resource-preserving logic is determined by the following:

- ▷ Instantiation of the language parameters $\mathbf{B}$, $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$;
- ▷ An equational specification $\mathcal{E}$ over the signature $\{+_i\}_{i\in\mathcal{I}}$;
- ▷ Two sets of indices $\{i_j\}_{j\in\mathcal{J}}, \{i_k\}_{k\in\mathcal{K}} \subseteq \mathcal{I}$;
- ▷ Two sets of equational specifications $\{\mathcal{E}_j\}_{j\in\mathcal{J}}$ and $\{\mathcal{E}_k\}_{k\in\mathcal{K}}$, where $\mathcal{E}_l$ is specified over the signature $\{+_{i_l}\}$ ($l \in \mathcal{J}\cup\mathcal{K}$).

Of course, all equational specifications occurring in the above list are required to be resource-preserving. The operator $+$ is intended as a generic one, which is to be replaced by a specific connective of the language on each separate occasion. We will write $\mathcal{E}^*$ for the equational specification obtained by substituting $*$ for $+$ in $\mathcal{E}$, but will drop this superscript when it is clear from the context. $(\mathcal{E}_j)_{\Delta_j}$ will be abbreviated as $\mathcal{E}_{\Delta_j}$, and $(\mathcal{E}_k)_{\nabla_k}$ as $\mathcal{E}_{\nabla_k}$.

Henceforth, we assume that we are dealing with a fixed resource-preserving logic $\mathcal{L}$.

2.4 Proof system

For $\mathcal{L}$ we have the following rules of inference:

$$\mathsf{A} \vdash \mathsf{A}$$

$$\frac{\Gamma\vdash \mathsf{A}\quad \Delta(\mathsf{B})\vdash \mathsf{C}}{\Delta[(\mathsf{B}/_i\mathsf{A})\circ_i\Gamma]\vdash \mathsf{C}}\,[/_i\mathrm{L}] \qquad \frac{\Gamma\circ_i \mathsf{A}\vdash \mathsf{B}}{\Gamma\vdash \mathsf{B}/_i\mathsf{A}}\,[/_i\mathrm{R}]$$

$$\frac{\Gamma\vdash \mathsf{A}\quad \Delta(\mathsf{B})\vdash \mathsf{C}}{\Delta[\Gamma\circ_i(\mathsf{A}\backslash_i\mathsf{B})]\vdash \mathsf{C}}\,[\backslash_i\mathrm{L}] \qquad \frac{\mathsf{A}\circ_i\Gamma\vdash \mathsf{B}}{\Gamma\vdash \mathsf{A}\backslash_i\mathsf{B}}\,[\backslash_i\mathrm{R}]$$

$$\frac{\Gamma_1\vdash \mathsf{Q}_l\quad \Gamma_2\vdash \mathsf{Q}_l\quad \Gamma[\mathsf{Q}_l]\vdash \mathsf{A}}{\Gamma[\Gamma_1\circ_{i_l}\Gamma_2]\vdash \mathsf{A}}\,[\mathsf{Q}_l]$$

$$\frac{\Gamma[\mathsf{A}]\vdash \mathsf{B}}{\Gamma[\Delta_j\mathsf{A}]\vdash \mathsf{B}}\,[\Delta_j\mathrm{L1}] \qquad \frac{\Gamma[\mathsf{Q}_j]\vdash \mathsf{B}}{\Gamma[\Delta_j\mathsf{A}]\vdash \mathsf{B}}\,[\Delta_j\mathrm{L2}]$$

$$\frac{\Gamma[\mathsf{A}]\vdash \mathsf{B}}{\Gamma[\nabla_k\mathsf{A}]\vdash \mathsf{B}}\,[\nabla_k\mathrm{L1}] \qquad \frac{\Gamma[\mathsf{Q}_k]\vdash \mathsf{B}}{\Gamma[\nabla_k\mathsf{A}]\vdash \mathsf{B}}\,[\nabla_k\mathrm{L2}]$$

$$\frac{\Gamma\vdash \mathsf{A}\quad \Gamma\vdash \mathsf{Q}_j}{\Gamma\vdash \Delta_j\mathsf{A}}\,[\Delta_j\mathrm{R}] \qquad \frac{\Gamma\vdash \mathsf{A}\quad \Gamma\vdash \mathsf{Q}_k}{\Gamma\vdash \nabla_k\mathsf{A}}\,[\nabla_k\mathrm{R}]$$

$$\frac{\Gamma\vdash \mathsf{A}}{\Delta\vdash \mathsf{A}}\updownarrow[\mathcal{E}] \qquad \frac{\Gamma\vdash \mathsf{A}\quad \Gamma_1\vdash \mathsf{Q}_l\ \ \ldots\ \ \Gamma_n\vdash \mathsf{Q}_l}{\Delta\vdash \mathsf{A}}\updownarrow[\mathcal{E}_l]$$

$$\frac{\Gamma\vdash \mathsf{A}\quad \Delta[\mathsf{A}]\vdash \mathsf{B}}{\Delta[\Gamma]\vdash \mathsf{B}}\,[\mathrm{Cut}]$$

In these rules i, j and k range over $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$, respectively, and l ranges over $\mathcal{J}\cup\mathcal{K}$. As before, a $\updownarrow$ indicates that we have a two-way inference rule. The $[\mathcal{E}_{(l)}]$-rule schemata are subject to the following condition: there exist an equation $s = t \in \mathcal{E}^{\circ}_{(l)}$ and a substitution $\sigma : \mathcal{V} \to \mathbf{T}$ such that Δ can be obtained from Γ by replacing a substructure s^{σ} of Γ with t^{σ}. On $[\mathcal{E}_l]$ we put the further restriction that the Γ_i's are exactly the elementary substructures of s^{σ}. For example, for $\mathcal{E}_j = \{x+y = y+x\}$ we would obtain the following rule:

$$\frac{\Gamma_1\vdash \mathsf{Q}_j\quad \Gamma_2\vdash \mathsf{Q}_j\quad \Gamma[\Gamma_1\circ_{i_j}\Gamma_2]\vdash \mathsf{A}}{\Gamma[\Gamma_2\circ_{i_j}\Gamma_1]\vdash \mathsf{A}}\updownarrow[\mathcal{E}_j]$$

$$
\dfrac{\mathsf{NP} \vdash \mathsf{NP} \qquad \dfrac{\mathsf{NP\backslash S} \vdash \mathsf{NP\backslash S} \qquad \dfrac{\mathsf{NP} \vdash \mathsf{NP} \quad \mathsf{S} \vdash \mathsf{S}}{\mathsf{NP}, \mathsf{NP\backslash S} \vdash \mathsf{S}}\,[\backslash L]}{\mathsf{NP}, \mathsf{NP\backslash S}, \mathsf{(NP\backslash S)\backslash(NP\backslash S)} \vdash \mathsf{S}}\,[\backslash L]}{\dfrac{\mathsf{NP}, \mathsf{NP\backslash S/NP}, \mathsf{NP}, \mathsf{(NP\backslash S)\backslash(NP\backslash S)} \vdash \mathsf{S}}{\dfrac{\mathsf{NP}, \mathsf{NP\backslash S/NP}, \nabla_P\mathsf{NP}, \mathsf{(NP\backslash S)\backslash(NP\backslash S)} \vdash \mathsf{S}}{\dfrac{\mathsf{NP}, \mathsf{NP\backslash S/NP}, \mathsf{(NP\backslash S)\backslash(NP\backslash S)}, \nabla_P\mathsf{NP} \vdash \mathsf{S}}{\mathsf{NP}, \mathsf{NP\backslash S/NP}, \mathsf{(NP\backslash S)\backslash(NP\backslash S)} \vdash \mathsf{S}/\nabla_P\,\mathsf{NP}}\,[/R]}\,[\mathcal{E}_{\nabla_P}]}\,[\nabla_P L1]}\,[/L]
$$

$$
\dfrac{\mathsf{NP}, \mathsf{NP\backslash S/NP}, \mathsf{(NP\backslash S)\backslash(NP\backslash S)} \vdash \mathsf{S}/\nabla_P\,\mathsf{NP} \qquad \mathsf{REL} \vdash \mathsf{REL}}{\mathsf{REL}/(\mathsf{S}/\nabla_P\,\mathsf{NP}), \mathsf{NP}, \mathsf{NP\backslash S/NP}, \mathsf{(NP\backslash S)\backslash(NP\backslash S)} \vdash \mathsf{REL}}\,[/L]
$$

Figure 1

$$
\dfrac{\mathsf{N} \vdash \mathsf{N} \qquad \dfrac{\mathsf{N} \vdash \mathsf{N} \quad \mathsf{NP} \vdash \mathsf{NP}}{(\mathsf{NP/N}) \circ \mathsf{N} \vdash \mathsf{NP}}\,[/L]}{\dfrac{(\mathsf{NP/N}) \circ (\mathsf{N} \circ (\mathsf{N\backslash N})) \vdash \mathsf{NP}}{\dfrac{(\mathsf{NP/N}) \circ (\mathsf{N} \circ \nabla_A(\mathsf{N\backslash N})) \vdash \mathsf{NP}}{\dfrac{((\mathsf{NP/N}) \circ \mathsf{N}) \circ \nabla_A(\mathsf{N\backslash N}) \vdash \mathsf{NP}}{(\mathsf{NP/N}) \circ \mathsf{N} \vdash \mathsf{NP}/\nabla_A\,(\mathsf{N\backslash N})}\,[/R]}\,[\mathcal{E}_{\nabla_A}]}\,[\nabla_A L1]}\,[\backslash L]
$$

$$
\dfrac{\dfrac{\mathsf{N} \vdash \mathsf{N} \quad \mathsf{NP} \vdash \mathsf{NP}}{(\mathsf{NP/N}) \circ \mathsf{N} \vdash \mathsf{NP}}\,[/L] \qquad \dfrac{(\mathsf{NP/N}) \circ \mathsf{N} \vdash \mathsf{NP}/\nabla_A\,(\mathsf{N\backslash N}) \qquad \mathsf{NP} \vdash \mathsf{NP}}{((\mathsf{NP/N}) \circ \mathsf{N}) \circ ((\mathsf{NP}/\nabla_A\,(\mathsf{N\backslash N}))\backslash\mathsf{NP}) \vdash \mathsf{NP}}\,[\backslash L]}{((\mathsf{NP/N}) \circ \mathsf{N}) \circ ((((\mathsf{NP}/\nabla_A\,(\mathsf{N\backslash N}))\backslash\mathsf{NP})/\mathsf{NP}) \circ ((\mathsf{NP/N}) \circ \mathsf{N})) \vdash \mathsf{NP}}\,[/L]
$$

Figure 2

2.5 Some sample applications

We will address the logical aspects of the calculi defined in the last section shortly, but first we pause for a brief intermezzo, illustrating how they are applied in linguistic practice.

As our first example we look at how the Lambek calculus deals with extraction. Suppose we have the following type assignments:

John, Mary	:	NP
loves	:	NP\S/NP
madly	:	(NP\S)\(NP\S)

We would like to find type assignments to *who* such that we can derive type REL for the following phrases:

1. *who John loves*
2. *who loves Mary*
3. *who John loves madly*

As is easily seen, assignment of REL/(S/NP) to *who* works for the first sentence, while REL/(NP\S) is the appropriate type to assign to *who* to get the second case right. However, the third case can't be done in the Lambek calculus, since we have no way of referring to gaps occuring *inside* larger constituents; we only have access to the periphery. This can be handled by adding a permutation modality ∇_P and assigning to *who* the type REL/(S/ ∇_P NP) to *who*. This single type assignment works for all three cases. For the third sentence, this is worked out in Figure 1.

As a second example, consider the following noun phrase:

the man at the desk

For the nouns and the determiner we make the usual type assignments:

the	:	NP/N
man, desk	:	N

From a prosodic point of view, *at* should be assigned type (N\N)/NP. However, semantically *at* combines not just with the noun it modifies, but with the entire noun phrase headed by that noun. Moortgat & Morrill [1992] show how both these desiderata can be fulfilled. First, the type assignment to *at* is lifted to ((NP/(N\N))\NP)/NP in order to force the required semantic combination. This is not the end of the story, because due to the non-associativity of the prosodic algebra we still can't derive a type NP for *the man at the desk*. To enable this, they add a structural modality ∇_A to the type assignment for *at* to make it ((NP/ ∇_A (N\N))\NP)/NP, after which things work out nicely, as is shown by the derivation in Figure 2.

2.6 Cut-elimination and the subformula property

Before turning to the semantics of $\mathcal{L}$ we will prove the Cut-elimination theorem and subformula property for it, since the latter is essential for the completeness proof, and a corollary to the former.

As we remarked earlier, our proof rules are adapted from [Venema, 1993b]. Therefore, we can refer the reader to that paper for most of the Cut-elimination proof. The only notable difference between both systems lies in the structural rules they allow. Note that resource-preservation implies that for any $[\mathcal{E}_{(j)}]$-inference we have the following two simple but important properties (where the complexity of a type is defined as the number of connectives occurring in it):

1. Each type occurring in Γ occurs also in Δ, and vice versa;
2. The complexity of Γ equals that of Δ.

Therefore, in the case of an $[\mathcal{E}_{(l)}]$-inference, we can always move [Cut] upwards like this is done in Venema's paper, and thus obtain an application of [Cut] of lower degree. Hence, [Cut] is eliminable from $\mathcal{L}$.

The subformula property says that any provable sequent has a proof in which only subformulas of that sequent occur. Under the proviso that Q_j is considered a subtype of $\Delta_j A$, and Q_k of $\nabla_k A$, the subformula property follows from Cut-elimination, since in each inference rule other than [Cut], the premises are made up of subformulas of the conclusion.

Let $\mathcal{L}_\bullet$ be the logic obtained from $\mathcal{L}$ by adding a set of product connectives $\{\bullet_i\}_{i\in\mathcal{I}}$ to the language, and the following inference rules to the proof system:

$$\frac{\Gamma \circ_i \Delta \vdash A}{\Gamma \bullet_i \Delta \vdash A}\,[\bullet_i L] \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma \circ_i \Delta \vdash A \bullet_i B}\,[\bullet_i R]$$

Like $\mathcal{L}$, the system $\mathcal{L}_\bullet$ enjoys Cut-elimination and the subformula property. Note that this implies that if an $\mathcal{L}$-sequent is $\mathcal{L}_\bullet$-derivable, then it is $\mathcal{L}$-derivable. This property will be used several times in the course of the completeness proof.

Now consider a naive top-down[2] proof search strategy. At every step, we have a finite choice of possible applications of an inference rule, and every such application either removes a connective occurence, thus diminishing the complexity of the sequent to be proved, or rewrites the sequent's antecedent to a term of equal complexity. Therefore, if we make sure that a search path is relinquished whenever a sequent reappears on it (which prevents the procedure from entering into an infinite loop), the proof search tree will be finite. This implies that the calculus is decidable.

2.7 Semantics

The basis for any model of $\mathcal{L}$ is a $(\Sigma, \mathcal{E})$-algebra $\mathcal{A}$, where $\Sigma = \{+_i\}_{i\in\mathcal{I}}$ and the product operation interpreting $\circ_i$ is denoted as $\cdot_i$. We say that $\mathcal{S} \subseteq \mathcal{A}$ is an *$\mathcal{E}_j$-subalgebra* of $\mathcal{A}$ if it is closed under $\cdot_{i_j}$, and $s^\sigma = t^\sigma$ whenever $s = t \in \mathcal{E}_j$ and $\sigma : \mathcal{V} \to \mathcal{S}$. An *easy $\mathcal{E}_k$-subalgebra* of $\mathcal{A}$ is a subset of $\mathcal{A}$ that is closed under $\cdot_{i_k}$, and such that $s^\sigma = t^\sigma$ whenever $s = t \in \mathcal{E}_k$ and $\sigma : \mathcal{V} \to \mathcal{A}$ assigns an element of $\mathcal{S}$ to at last one of the variables occurring in the equation. A *model* for $\mathcal{L}$ is a 4-tuple $\langle \mathcal{A}, \{\mathcal{A}_j\}_{j\in\mathcal{J}}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, [\![.]\!]\rangle$ such that:

- ▷ $\mathcal{A}$ is a $(\Sigma, \mathcal{E})$-algebra;
- ▷ $\mathcal{A}_j$ is an $\mathcal{E}_j$-subalgebra of $\mathcal{A}$ ($j \in \mathcal{J}$);
- ▷ $\mathcal{A}_k$ is an easy $\mathcal{E}_k$-subalgebra of $\mathcal{A}$ ($k \in \mathcal{K}$);
- ▷ $[\![.]\!]$ is a function $\mathbf{B} \to \mathcal{P}(\mathcal{A})$.

Here, $\mathcal{P}(\mathcal{A})$ denotes the set of all subsets of $\mathcal{A}$. The interpretation function $[\![.]\!]$ is extended to arbitrary types and structures as follows:

- ▷ $[\![Q_l]\!] = \mathcal{A}_l$ ($l \in \mathcal{J} \cup \mathcal{K}$)
- ▷ $[\![B/_iA]\!] = \{c \in \mathcal{A} \mid \forall a \in [\![A]\!] : c \cdot_i a \in [\![B]\!]\}$
- ▷ $[\![A\backslash_iB]\!] = \{c \in \mathcal{A} \mid \forall a \in [\![A]\!] : a \cdot_i c \in [\![B]\!]\}$
- ▷ $[\![A \circ_i B]\!] = \{c \in \mathcal{A} \mid \exists a \in [\![A]\!], b \in [\![B]\!] : c = a \cdot_i b\}$

A sequent $\Gamma \vdash A$ is said to be *valid* with respect to a given model, if $[\![\Gamma]\!] \subseteq [\![A]\!]$. A sequent is *generally valid* if it is valid in all models. The proof system is said to be *sound* with respect to the semantics if all derivable sequents are generally valid. It is *complete* if the converse holds, i.e. if all generally valid sequents are derivable.

2.8 Soundness and completeness

As usual, the soundness proof boils down to a straightforward induction on the length of a derivation, and we omit it.

For completeness, we start by defining the *canonical model* $\mathcal{M}$. Its carrier is the set $\mathbf{S}/\!\equiv$, where $\equiv$ is the equivalence relation defined by $\Gamma \equiv \Delta$ iff $\forall A : \Gamma \vdash A \Leftrightarrow \Delta \vdash A$. The $\equiv$-equivalence class containing Γ will be denoted as $[\Gamma]$. On the set $\mathbf{S}/\!\equiv$ we define products $\cdot_i$ ($i \in \mathcal{I}$) by stipulating that $[\Gamma] \cdot_i [\Delta] = [\Gamma \circ_i \Delta]$. We need to prove that this is well-defined. So suppose $\Gamma \equiv \Gamma'$, $\Delta \equiv \Delta'$ and $\Gamma \circ_i \Delta \vdash A$. For a structure Θ, let $\Theta^\bullet$ be the $\mathcal{L}_\bullet$-type obtained from Θ by replacing each $\circ_i$ with $\bullet_i$. The sequent $\Theta^\bullet \vdash A$ can be derived from $\Theta \vdash A$ by a sequence of $[\bullet L]$-rules. By definition of $\equiv$ we know that $\Gamma' \vdash \Gamma^\bullet$ and $\Delta' \vdash \Delta^\bullet$. Now, $\Gamma' \circ_i \Delta' \vdash A$ by the derivation below:

$$\cfrac{\cfrac{\cfrac{\Gamma \circ_i \Delta \vdash A}{\Gamma^\bullet \circ_i \Delta^\bullet \vdash A}\,[\bullet L]^* \quad \Gamma' \vdash \Gamma^\bullet}{\Gamma' \circ_i \Delta^\bullet \vdash A}\,[\text{Cut}] \quad \Delta' \vdash \Delta^\bullet}{\Gamma' \circ_i \Delta' \vdash A}\,[\text{Cut}]$$

Evidently, $\mathcal{M} = (\mathbf{S}/\!\equiv, \{\cdot_i\}_{i\in\mathcal{I}})$ is a $(\Sigma, \mathcal{E})$-algebra.

Next, we define $\mathcal{M}_l = \{[\Gamma] \mid \Gamma \vdash Q_l\}$ ($l \in \mathcal{J} \cup \mathcal{K}$). It must be shown that these have the desired properties. Since it would be notationally awkward to have to refer to an arbitrary equational specification, we do this by means of an example. Let

[2] Note that we use the term *top-down* in the usual sense, i.e. for a proof search procedure that works back from the goal to the axioms. Visually, top-down proofs actually proceed bottom-up!

$$\dfrac{\dfrac{\dfrac{Q_j \vdash Q_j}{\Delta_j\Gamma^\bullet_{1,2} \vdash Q_j}\,[\Delta_j L2] \quad \dfrac{\dfrac{\Gamma_1 \circ_{i_j} \Gamma_2 \vdash A}{\Gamma^\bullet_1 \circ_{i_j} \Gamma^\bullet_2 \vdash A}\,[\bullet L]}{\Delta_j\Gamma^\bullet_1 \circ_{i_j} \Delta_j\Gamma^\bullet_2 \vdash A}\,[\Delta_j L1]}{\Delta_j\Gamma^\bullet_2 \circ_{i_j} \Delta_j\Gamma^\bullet_1 \vdash A}\,[\mathcal{E}_{\Delta_j}] \quad \dfrac{\Gamma_1 \vdash \Gamma^\bullet_1 \quad \Gamma_1 \vdash Q_j}{\Gamma_1 \vdash \Delta_j\Gamma^\bullet_1}\,[\Delta_j R]}{\Gamma_2 \circ_{i_j} \Delta_j\Gamma^\bullet_1 \vdash A}\,[\text{Cut}] \quad \dfrac{\Gamma_2 \vdash \Gamma^\bullet_2 \quad \Gamma_2 \vdash Q_j}{\Gamma_2 \vdash \Delta_j\Gamma^\bullet_2}\,[\Delta_j R]$$

$$\overline{\qquad \Gamma_2 \circ_{i_j} \Gamma_1 \vdash A \qquad}\ [\text{Cut}]$$

Figure 3

$\mathcal{E}_{\Delta_j} = \{\Delta_j x +_{i_j} \Delta_j y = \Delta_j y +_{i_j} \Delta_j x\}$. Supposing that $[\Gamma_1], [\Gamma_2] \in \mathcal{M}_{\Delta_j}$ we must prove that $[\Gamma_1] \cdot_{i_j} [\Gamma_2] = [\Gamma_2] \cdot_{i_j} [\Gamma_1]$, i.e. that $\forall A : \Gamma_1 \circ_{i_j} \Gamma_2 \vdash A \Leftrightarrow \Gamma_2 \circ_{i_j} \Gamma_1 \vdash A$. This follows from the derivation in Figure 3. The proof for $\mathcal{M}_{\nabla_k}$ is similar.

Finally, we set $[\![B]\!] = \{[\Gamma] \mid \Gamma \vdash B\}$ for $B \in \mathbf{B}$, which completes our definition of the canonical model.

We proceed to prove the so-called *canonical lemma*:

Lemma

$[\![T]\!] = \{[\Gamma] \mid \Gamma \vdash T\}$ for all $T \in \mathbf{T}$.

Proof

We prove this by induction on the complexity of the type T.

- ▹ For basic types T it is true by the definition of $[\![\cdot]\!]$;
- ▹ For Q_l ($l \in \mathcal{J} \cup \mathcal{K}$) it is true by the definition of $\mathcal{M}_l$;
- ▹ For $T = B/_iA$:
  1. First, suppose $[\Gamma] \in [\![T]\!] = [\![B/_iA]\!]$. Then for any $[\Delta] \in [\![A]\!]$ we have that $[\Gamma] \cdot_i [\Delta] = [\Gamma \circ_i \Delta] \in [\![B]\!]$. By the induction hypothesis we deduce from this that $\Gamma \circ_i \Delta \vdash B$. In particular, since $[A] \in [\![A]\!]$, we have that $\Gamma \circ_i A \vdash B$, whence, by $[/_iR]$, it follows that $\Gamma \vdash B/_iA$.
  2. Conversely, suppose that $\Gamma \vdash B/_iA$, and let $[\Delta] \in [\![A]\!]$. Then, by the induction hypothesis, $\Delta \vdash A$. We now have the following derivation:

$$\dfrac{\Delta \vdash A \quad \dfrac{\Gamma \vdash B/_iA \quad \dfrac{A \vdash A \quad B \vdash B}{(B/_iA) \circ_i A \vdash B}\,[/_iL]}{\Gamma \circ_i A \vdash B}\,[\text{Cut}]}{\Gamma \circ_i \Delta \vdash B}\,[\text{Cut}]$$

  From this we conclude by the induction hypothesis that $[\Gamma \circ_i \Delta] = [\Gamma] \cdot_i [\Delta] \in [\![B]\!]$ for all $[\Delta] \in [\![A]\!]$. That is, $[\Gamma] \in [\![B/_iA]\!]$, and we're done.

  For the other binary connectives, the proof is similar.
- ▹ For $T = \Delta_jA$:
  1. First, suppose $[\Gamma] \in [\![\Delta_jA]\!] = [\![A]\!] \cap \mathcal{M}_j$. Then, by the induction hypothesis, $\Gamma \vdash A$. Also, by the definition of $\mathcal{M}_j$, $\Gamma \vdash Q_j$. Applying the $[\Delta_jR]$-rule two these two sequents, we find that $\Gamma \vdash \Delta_jA$.
  2. Conversely, suppose $\Gamma \vdash \Delta_jA$. Then $\Gamma \vdash A$:

$$\dfrac{\Gamma \vdash \Delta_jA \quad \dfrac{A \vdash A}{\Delta_jA \vdash A}\,[\Delta_jL1]}{\Gamma \vdash A}\,[\text{Cut}]$$

  From this we conclude by the induction hypothesis that $[\Gamma] \in [\![A]\!]$. Also, $\Gamma \vdash Q_j$:

$$\dfrac{\Gamma \vdash \Delta_jA \quad \dfrac{Q_j \vdash Q_j}{\Delta_jA \vdash Q_j}\,[\Delta_jL2]}{\Gamma \vdash Q_j}\,[\text{Cut}]$$

  From this we find by the definition of $\mathcal{M}_j$ that $[\Gamma] \in [\![Q_j]\!] = \mathcal{M}_j$. So $[\Gamma] \in [\![A]\!] \cap [\![Q_j]\!] = [\![\Delta_jA]\!]$.

  For ∇_k, the proof is similar.

Now suppose that the sequent $\Gamma \vdash A$ is not derivable. Then in the canonical model we have, by the lemma we just proved, that $[\Gamma] \notin [\![A]\!]$. Since $[\Gamma] \in [\![\Gamma]\!]$, this implies that $[\![\Gamma]\!] \not\subseteq [\![A]\!]$. That is, $\Gamma \vdash A$ is not valid in the canonical model, and hence is not generally valid. □

3 Further research

It will not have escaped the reader's attention that we have failed to include the set of product connectives $\{\bullet_i\}_{i \in \mathcal{I}}$ in the language of the resource-preserving logics. The reason for this is that a completeness proof along the above lines runs into problems for such extended logics. This is already the case for the full Lambek calculus. Buszkowski [1986] presents a rather complicated completeness proof for that logic. It remains to be seen whether his approach also works in the present setting.

Although we've tried to give a liberal definition of what constitutes a resource-preserving logic, some choices had to be made in order to keep things manageable. There is room for alternative definitions, especially concerning the interaction of the modalities with the different product operators. It would seem to be worthwile to study some of the systems that have occurred in practice in detail on the basis of the ideas presented in this paper.

Finally, it is important to realize that we limited ourselves to resource-preserving logics in order to obtain relatively easy proofs of Cut-elimination and decidability. Since such results tend also to hold for many systems with rules that are not resource-preserving, such as weakening and contraction, it is probably possible to characterize a larger class of equational theories for which these properties can be proved. We hope to address this point on a later occassion.

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