Grammar Equations

Bob Coecke
Oxford-based QNLP-team
Cambridge Quantum Computing Ltd.
bob.coecke@cambridgequantum.com

Vincent Wang
Department of CS
University of Oxford
vincent.wang@stcatz.ox.ac.uk

Abstract

Diagrammatically speaking, grammatical calculi such as pregroups provide wires between words in order to elucidate their interactions, and this enables one to verify grammatical correctness of phrases and sentences. In this paper we also provide wirings within words. This will enable us to identify grammatical constructs that we expect to be either equal or closely related. Hence, our work paves the way for a new theory of grammar, that provides novel 'grammatical truths'. We give a nogo-theorem for the fact that our wirings for words make no sense for preordered monoids, the form which grammatical calculi usually take. Instead, they require diagrams – or equivalently, (free) monoidal categories.

1 Introduction

Grammatical calculi (Lambek, 1958; Grishin, 1983; Lambek, 1999) enable one to verify grammatical correctness of sentences. However, there are certain grammatical constructs that we expect to be closely related, if not the same, but which grammatical calculi fail to identify. We will focus on pregroups (Lambek, 2008), but the core ideas of this paper extend well beyond pregroup grammars, including CCGs (Steedman, 1987), drawing on the recent work in (Yeung and Kartsaklis, 2021) that casts CCGs as augmented pregroups.

In this paper we both modify and extend grammatical calculi, by providing so-called 'internal wirings' for a substantial portion of English. Diagrammatically speaking, while grammatical calculi provide wires between words in order to elucidate their interactions, we also provide wirings within words. For example, a *pregroup diagram* for the phrase:
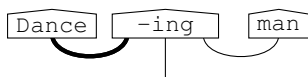
will become:
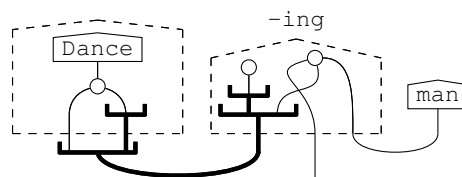
We show how these additional wirings enable one to identify grammatical constructs that we expect to be closely related. Providing these internal wirings in particular involves decomposing basic types like sentence-types over noun-types, and this decomposition may vary from sentence to sentence. Hence our refinement of grammar-theory also constitutes a departure from some of the practices of traditional grammatical calculi.

Additional structure for grammatical calculi was previously introduced by providing semantics to certain words, for example, quantifiers within Montague semantics (Montague, 1973). This is not what we do. We strictly stay within the realm of grammar, and grammar only. Hence, our work paves the way for a new theory of grammar, that provides novel 'grammatical truths'.

Usually grammatical calculi take the form of preordered monoids (Coecke, 2013). However, the internal wirings cannot be defined at the poset level, for which we provide a nogo-theorem. Hence passing to the realm of diagrammatic representations – which correspond to proper free monoidal categories – is not just a convenience, but a necessity for this work. They moreover provide a clear insight in the flow of meanings.

Internal wirings were proposed within the DisCoCat framework (Coecke et al., 2010), for relative pronouns and verbs (Sadrzadeh et al., 2013, 2016; Grefenstette and Sadrzadeh, 2011; Kartsaklis and Sadrzadeh, 2014; Coecke et al., 2018; Coecke, 2019; Coecke and Meichanetzidis, 2020). They are made up of 'spiders' (a.k.a. certain

Frobenius algebras) (Coecke et al., 2013; Coecke and Kissinger, 2017). We point out a shortcoming of those earlier proposed internal wirings, and fix them by introducing a 'wrapping gadget', that forces certain wires to stay together. This reintroduces composite types such as sentence types.

What we present here is only part of the full story. For the latter we refer to a forthcoming much longer paper (Coecke and Wang), which besides providing many more internal wirings than given here, also uses them to provide bureaucracy-free grammar as circuits, the equivalence classes for the equations introduced here. These circuits also have direct practical applications within natural language processing – see e.g. (Coecke et al., 2020).
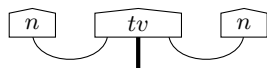
2 Statement of the problem

For our purposes, a pregroup has a set of 'basic types' $n, s, ...$ each of which admit left and right inverses ^{-1}n and n^{-1}. Each grammatical type is assigned a string of these, e.g. a transitive verb in English gets: $tv = {}^{-1}n \cdot s \cdot n^{-1}$. The inverses 'cancel out' from one direction:

$$n \cdot {}^{-1}n \to 1 \qquad n^{-1} \cdot n \to 1 \qquad (1)$$

A sentence is grammatical if when taking the string of all of its grammatical types, the inverses cancel to leave a special, 'final', basic type s (for sentence), like here for $n \cdot tv \cdot n$:

$$
\begin{aligned}
& n \cdot \left({}^{-1}n \cdot s \cdot n^{-1}\right) \cdot n \\
\overset{(assoc.)}{\to}\ & \left(n \cdot {}^{-1}n\right) \cdot s \cdot \left(n^{-1} \cdot n\right) \\
\overset{(1)}{\to}\ & 1 \cdot s \cdot 1 \\
\overset{(unit)}{\to}\ & s
\end{aligned}
$$

This calculation can be represented diagrammatically:

$$(2)$$

Now consider the following examples:

```
Alice likes the flowers that Bob gives
                 Claire
Bob gives Claire the flowers that Alice
                  likes
```

The pregroup diagrams now look as in Figure (1). Without any further context the factual data con-

veyed by these two sentences is the same.[1] How can we formally establish this connection between the two sentences?

3 Rewriting pregroup diagrams via internal wirings

What is needed are 'internal wirings' of certain words, that is, not treating these words as 'black boxes', but specifying what is inside, at least to some extent. Equationally speaking, they provide a congruence for pregroup diagrams, and we can establish equality by means of topological deformation.

For constructing these internal wirings we make use of 'spiders' (Coecke et al., 2013; Coecke and Kissinger, 2017) (a.k.a. Frobenius algebras (Carboni and Walters, 1987; Coecke and Paquette, 2011)). One can think of these spiders as a generalisation of wires to multi-wires, as rather than having two ends, they can have multiple ends. Still, all they do, like wires, is connect stuff, and when you connect connected stuff to other connected stuff (a.k.a. 'spider-fusion'):
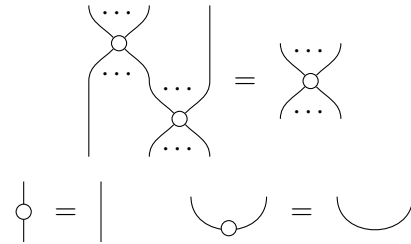
We presented internal wiring in terms of pregroup diagrams. This is because they do not make sense in terms of symbolic pregroups presented as preordered monoids:

Theorem 3.1. A pregroup with spiders is trivial. Concretely, given a preordered monoid $(X, \leq, \otimes)$ with unit 1, if for $x \in X$ there are spiders with x as its legs, then $x \simeq 1$.

Proof. Having spiders on x means that for all $j, k \in \mathbb{N}$ there exists:

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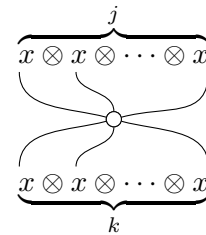
[1]Additional context could indicate a causal connection between the two parts of the sentence, which could result in the two sentences having different meanings – see (Coecke and Wang) for more details.
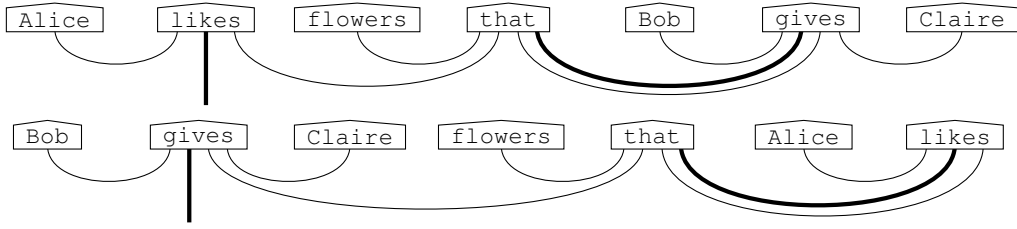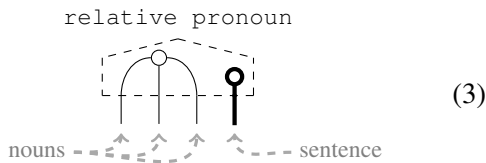
Figure 1

that is, we have $\bigotimes^j x \leq \bigotimes^k x$. So in particular, $x \leq 1$ and $1 \leq x$, so $x \simeq 1$. $\qquad\square$

Hence this paper requires diagrams in a fundamental manner.[2]

3.1 Internal wiring for relative pronouns

For relative pronouns we start with the internal wirings that were introduced in (Sadrzadeh et al., 2013, 2016):

$$(3)$$

Substituting this internal wiring in the pregroup diagrams we saw above: and permuting the boxes a bit, more specifically, swapping `Bob gives Claire` and `Alice likes` in the 2nd diagram, the two diagrams start to look a lot more like each other, as can be seen in figure 2. Their only difference is a twist which vanishes if we take spiders to be commutative,[3] and either a loose sentence-type wire coming out of the verb `likes` in the first diagram, versus coming out the verb `give` in the second diagram, the other verb having its sentence type deleted.
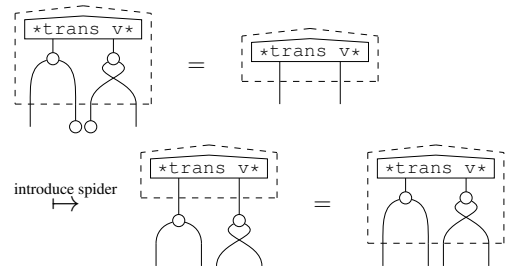
3.2 Internal wiring for verbs

The deleting of sentence-types of verbs:

$$(4)$$

<hr/>

[2] One SEMSPACE referee requested a category-theoretic generalisations of the above stated nogo-theorem. Such a generalisation has been provided on Twitter following our request (Hadzihasanovic, 2021). Our result should also not be confused with the (almost contradictory sounding) following one, which states that pregroups <u>are</u> spiders in the category of preordered relations (Pavlovic, 2021).

[3] Non-commutativity can be seen as a witness for the fact that within a broader context the two sentences may defer in meaning due to a potential causal connection between its two parts – see (Coecke and Wang) for more details.

by the internal wiring of relative pronouns seems to prevent us from bringing the diagrams of Figure (2) any closer to each other. However, this irreversibility does not happen for a particular kind of internal wiring for the verb (Grefenstette and Sadrzadeh, 2011; Kartsaklis and Sadrzadeh, 2014; Coecke, 2019; Coecke and Meichanetzidis, 2020), here generalised to the non-commutative case as demonstrated by the transitive verb in Figure (5). For transitive verbs in spider-form, if the sentence type gets deleted we can bring back the original form by copying the remaining wires:

So nothing was ever lost. To conclude, for the internal wiring of verbs proposed above, the copying and deleting spiders now guarantee that in (4) nothing gets lost.

3.3 Rewriting pregroup diagrams into each other

Introducing the internal wiring (5) and deleting all outputs, our example sentences now appear as in the first two diagrams of Figure (3). Except for the twist the two pregroup diagrams have become the same. As we have no outputs anymore, let's just stick in a copy-spider for all nouns, and then after fusing all deletes away, our sentence is transformed into the third diagram of figure 3.

The recipe we followed here is an instance of a general result that allows us to relate sentences for a substantial portion of English, by providing internal wirings for that fragment. In Section 5 we will provide internal wirings some grammatical word classes – in (Coecke and Wang) we provide a much larger catalog – that will generate
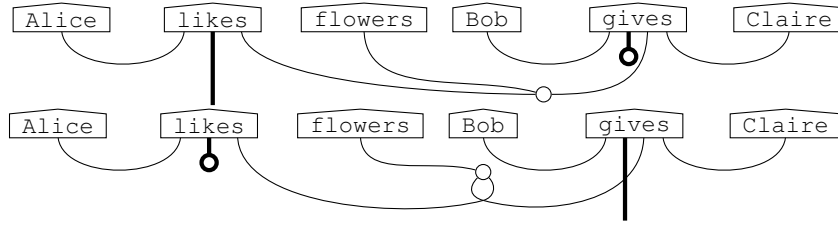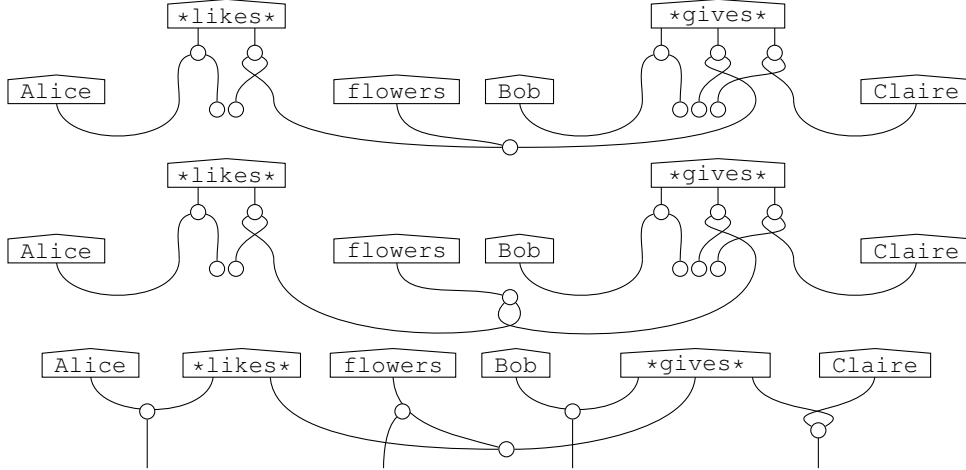
Figure 2

Figure 3

correspondences between grammatical constructs, just like the one established above. In Section 6 we provide some further examples of this. In (Coecke and Wang) we also provide a normal induced by grammar equations.

4 The wrapping gadget

Above in (5) we saw that sentence wires were decomposed into noun wires. However, for pregroup proofs it is important to know that those wires do belong together, so we need to introduce a tool that enables us to express that they belong together.

Definition 4.1. The wrapping gadget forces a number of wires to be treated as one, i.e. it wraps them, and is denoted as follows:

$$
\begin{array}{c|c|c}
Y_1 & Y_i & Y_N \\
\cdots & \cdots &
\end{array}
$$
$$
\left[\bigotimes_{i=1}^{N} Y_i \right]
$$

By unfolding we mean dropping the restrictions imposed by the wrapping gadget. Cups and spiders carry over to wrapped wires in the expected way, following the conventions of (Coecke and Kissinger, 2017).

In fact, in the case of relative pronouns simply wrapping the noun-wires making up the sentence type isn't enough, as the counterexample in Figure (4) shows.

5 Some more internal wirings

We now provide internal wirings for some grammatical word classes that feature in the examples of the next section. We distinguish between 'content words', like the verbs in (5), and 'functional words', like the relative pronouns in (7).

5.1 Content words

We provide internal wirings for intransitive and transitive verbs in Figure (5), and predicative and attributive adverbs for transitive verbs in Figure (6).

5.2 Functional words

We provide internal wirings for subject and object relative pronouns for intransitive verbs, and a passive-voice construction 'word' for transitive verbs in Figure (7).

6 Proof-of-concept

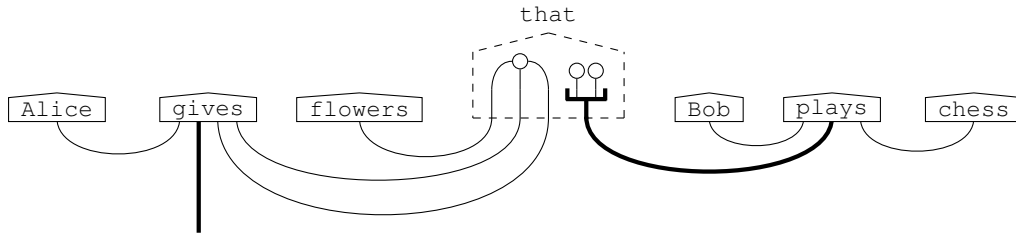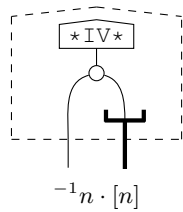We provide a number of examples of how the internal wirings proposed above enable us to re-

Figure 4: The deleting of the sentence type of `plays` belongs together with the noun-wire now connecting the relative pronoun with `gives`, like in Figure (1). This is enforced by the internal wiring of the object relative pronoun in Figure (7)

Intransitive Verb

$$^{-1}n \cdot [n]$$
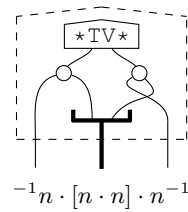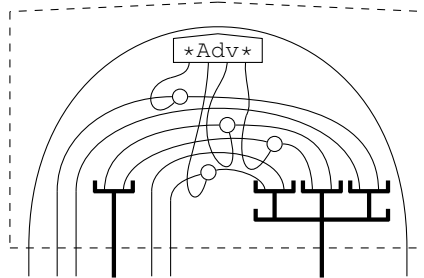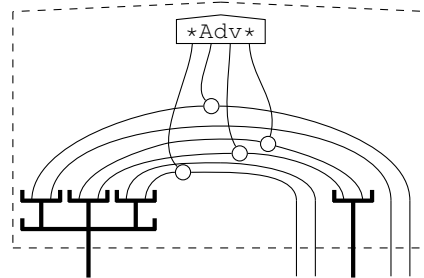
Transitive Verb

$$^{-1}n \cdot [n \cdot n] \cdot n^{-1}$$

Figure 5

Attributive Adverb$_{\text{TV}}$

$$^{-1}n \cdot n \cdot {}^{-1}n \cdot [n \cdot n] \cdot n^{-1} \cdot n \cdot [[n \cdot {}^{-1}n] \cdot [n \cdot n] \cdot [n^{-1} \cdot n]]^{-1} \cdot n$$
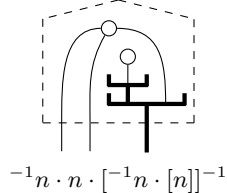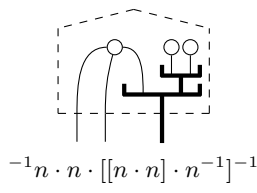
Predicative Adverb$_{\text{TV}}$

$$^{-1}[[n \cdot {}^{-1}n] \cdot [n \cdot n] \cdot [n^{-1} \cdot n]] \cdot n \cdot {}^{-1}n \cdot [n \cdot n] \cdot n^{-1} \cdot n$$

Figure 6

Sub. Rel. Pron.$_{\text{IV}}$

$$^{-1}n \cdot n \cdot [^{-1}n \cdot [n]]^{-1}$$

Ob. Rel. Pron.$_{\text{TV}}$

$$^{-1}n \cdot n \cdot [[n \cdot n] \cdot n^{-1}]^{-1}$$

Passive voice$_{\text{TV}}$

$$^{-1}[^{-1}n \cdot [n \cdot n] \cdot n^{-1}] \cdot {}^{-1}n \cdot [n \cdot n] \cdot n^{-1}$$
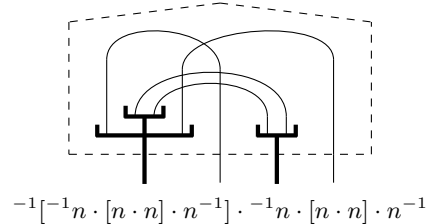
Figure 7

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late different grammatical constructs just as in the case of what the relative pronoun and verb internal wirings did for the sentences in Figure (1). We omit the pregroup typings, instead depicting the pregroup diagrams directly. Wrapping gadgets correspond to bracketing pregroup types together.

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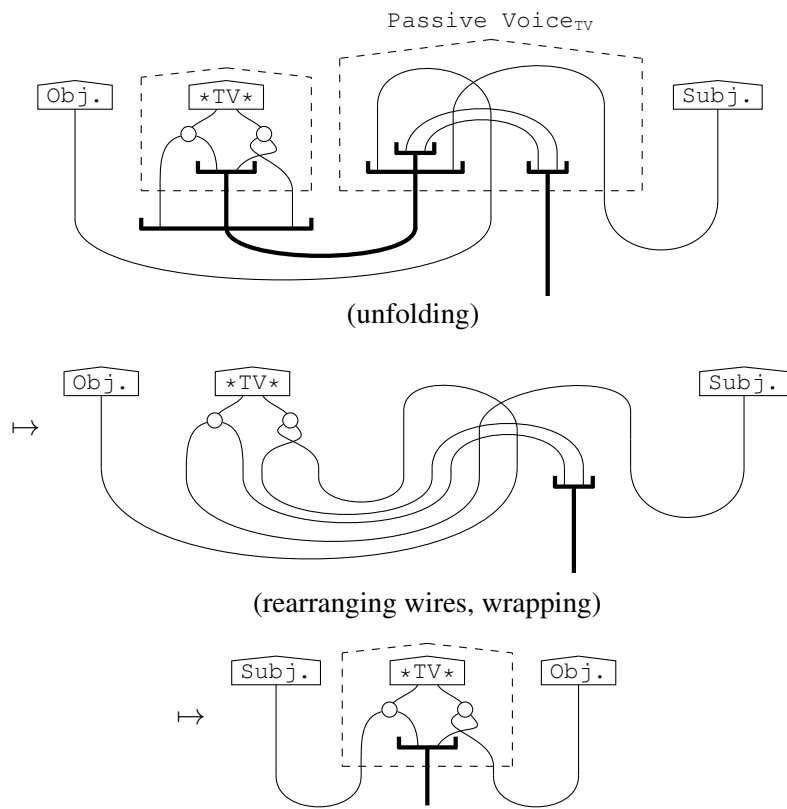R. Yeung and D. Kartsaklis. 2021. A CCG-based version of the DisCoCat framework. *Accepted for SEMSPACE*.

Figure 8: **We relate:** `Alice` $\underbrace{\text{is bored by}}_{\text{passive voice}}$ `the class` **to:** `The class bores Alice`
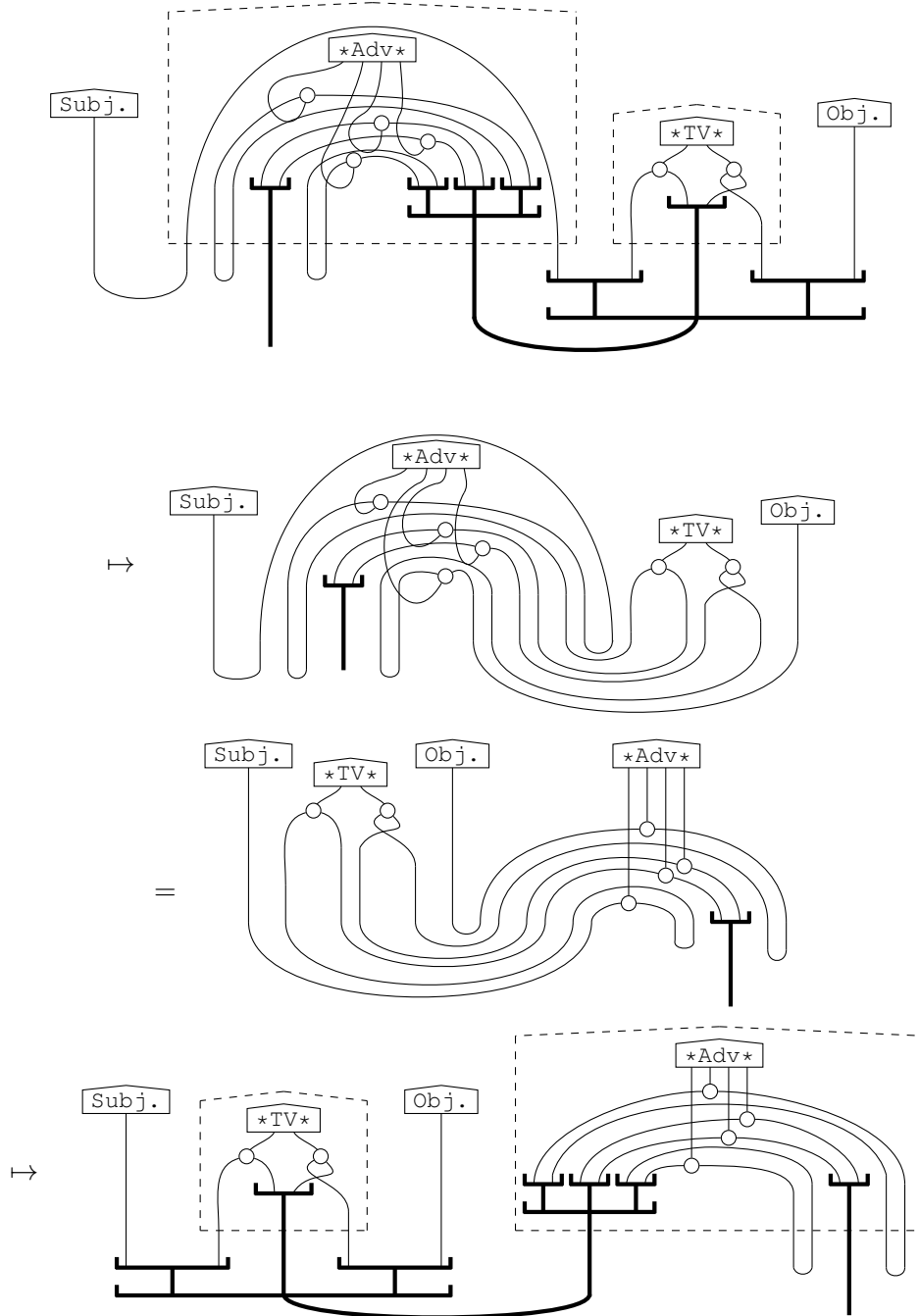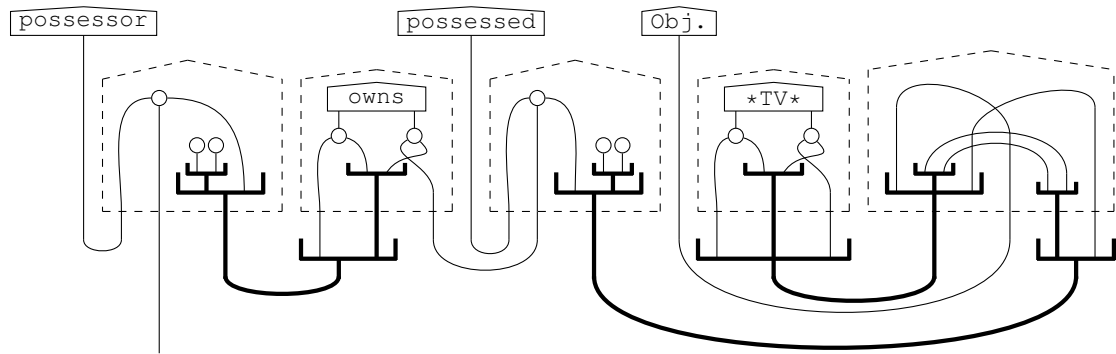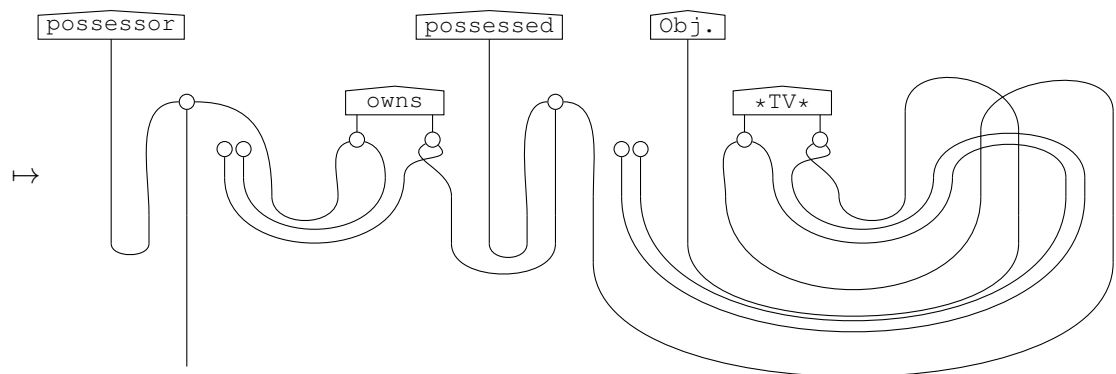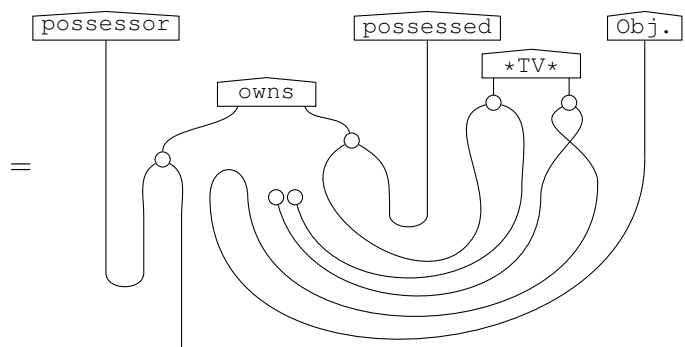
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Figure 9: **We relate:** `Alice washes Fido gently` **to:** `Alice gently washes Fido`

(unwrapping wires)

(dragging wires into place)

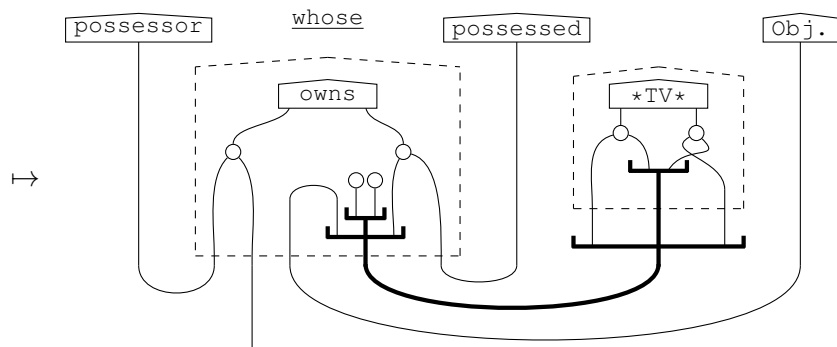(recovering a pregroup proof with bracketing)

Figure 10: **From:** `author that owns book that John (was) entertain(s) -ed (by)` **we derive a possessive relative pronoun:** `author` `whose` `book entertained John`
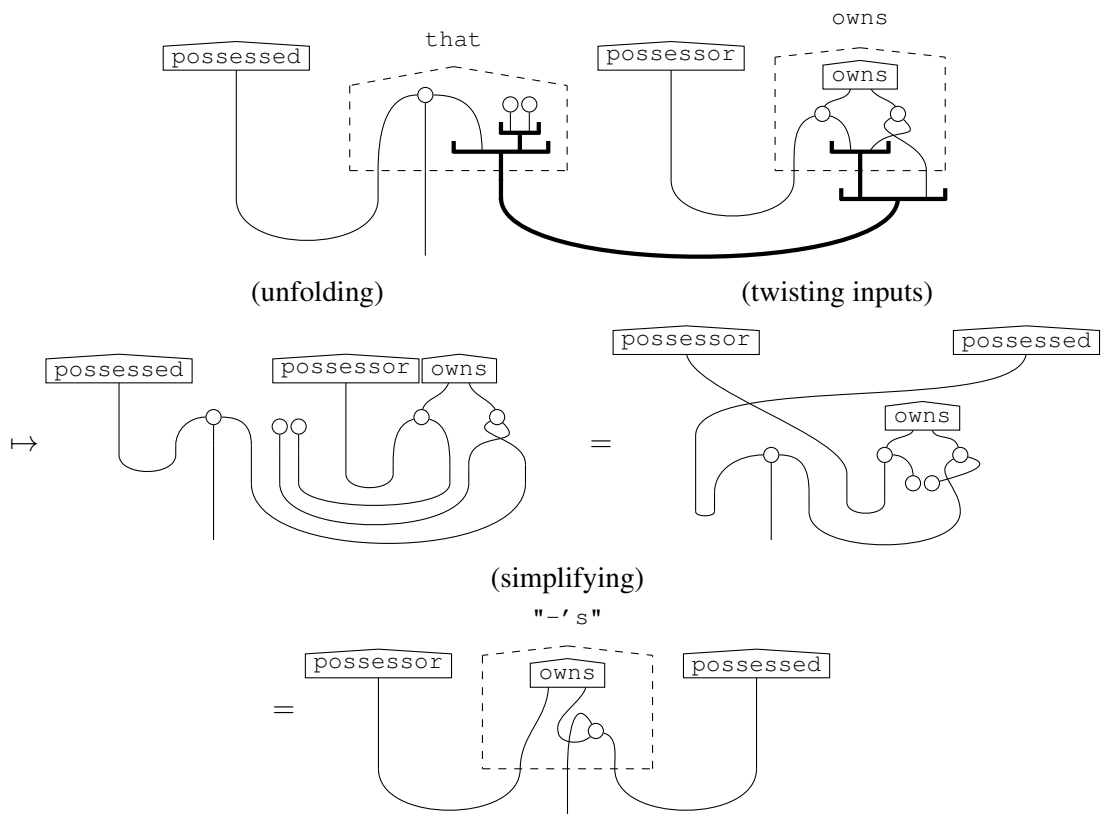
(unfolding) (twisting inputs)

(simplifying)

Figure 11: **From:** (possessed) that (possessor) owns **we derive the possessive modifier:** (possessor) 's (possessed)

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