It is not True that Transformers are Inductive Learners: Probing NLI Models with External Negation

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Abstract

NLI tasks necessitate a substantial degree of logical reasoning; as such, the remarkable performance of SoTA transformers on these tasks may lead us to believe that those models have learned to reason logically. The results presented in this paper demonstrate that (i) models fine-tuned on NLI datasets learn to treat external negation as a distractor, effectively ignoring its presence in hypothesis sentences; (ii) several near-SoTA encoder and encoder-decoder transformer models fail to inductively learn the law of the excluded middle for a single external negation prefix with respect to NLI tasks, despite extensive fine-tuning; (iii) those models which are able to learn the law of the excluded middle for a single prefix are unable to generalize this pattern to similar prefixes. Given the critical role of negation in logical reasoning, we may conclude from these findings that transformers do *not* learn to reason logically when fine-tuned for NLI tasks. Furthermore, these results suggest that transformers may not be able to inductively learn the role of negation with respect to NLI tasks, calling into question their capacity to fully acquire logical reasoning abilities.

1 Introduction

Natural language inference (NLI) tasks require detecting inferential relations between pairs of sentences (Fyodorov et al., 2000). For NLI datasets such as MultiNLI (MNLI; Williams et al., 2017) and Stanford NLI (SNLI; Bowman et al., 2015), the task proceeds as follows: given a pair of sentences (P, H), an NLI model must determine whether the *premise* P *entails* the *hypothesis* H, H *contradicts* P, or P and H are *neutral* with respect to one another (i.e. P does not entail H and H does not contradict P).

NLI tasks require logical reasoning capabilities that extend beyond basic linguistic competence (Richardson et al., 2020). For example, understanding that *"Jane is travelling to Algeria"* entails *"Jane is travelling to Africa"* requires mereological world knowledge (Hovda, 2009); an agent must know that Algeria is contained within Africa. To understand that *"Jane is traveling to Algeria"* does *not* entail *"Jane is traveling to Algiers"*, the agent must understand that Algiers is contained within Algeria, but that Algeria is not solely comprised of the city of Algiers.

Because of the considerable amount of reasoning that is required to accomplish NLI tasks, it is important to scrutinize the degree to which current NLI models are *actually* learning to reason logically. McCoy et al.'s (2019) findings suggest, for example, that even (then-)SoTA NLI models such as BERT (Devlin et al., 2019) adopt shallow, textual heuristics to achieve high-scoring results on the MNLI dataset, although the MNLI dataset itself is likely to be—at least partially—at fault (possibly because the provided training data is not sufficiently representative of the task; see the discussion in Section 2).

This paper investigates SoTA transformer (Vaswani et al., 2017) NLI models' ability to inductively learn the law of the excluded middle (LEM) with respect to *external negation* (negation that occurs externally to the proposition that is negated, e.g. *"<u>it is not true that</u> apples are red"*), in order to evaluate the degree to which they have learned to reason logically when performing NLI tasks. Using external negation, it is possible to automatically construct augmented challenge examples from the MNLI and SNLI datasets that modify the original examples' class labels in a predictable manner: given a premise, hypothesis, label triple (P, H, L), we generate an augmented example $(P, \neg H, L')$, where $L' =$ *neutral* if $L =$ *neutral*, $L' =$ *contradiction* if $L =$ *entailment*, and $L' =$ *entailment* if $L =$ *contradiction*.

Experiments 1 and 2 evaluate the NLI models' inductive learning capacity along two respective

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axes: Experiment 1 (Section 4) examines these models' ability to generalize double-negation cancellation to chains of repeated external negation prefixes longer than those seen during inoculation (∼fine-tuning; see Section 2 for an in-depth description), with respect to a single prefix string. We observe that NLI models struggle to learn this pattern inductively, with many unable to learn it at all. Experiment 2 (Section 5) evaluates the ability of those NLI models which were successfully able to learn LEM for a single external negation prefix to generalize this pattern to prefix strings not seen during fine-tuning. We find that those inoculated models suffer drastic decreases in performance when presented with unseen prefixes; the results of Experiment 3 (Section 6) indicate that this is due to catastrophic forgetting of the similarity between the prefix that they were inoculated against and other, highly similar prefixes.

The experimental results contained in this paper[1] indicate that transformer models do not learn to reason logically when fine-tuned on NLI datasets, lending further support to McCoy et al.'s (2019) hypothesis that they are instead learning to leverage shallow heuristics. In Section 3, we find evidence (Theorem 1) that this failure of transformer models to inductively learn LEM arises from deficiencies in their training procedure and/or the structure (or lack thereof) of their input data, rather than flaws inherent to transformer architectures themselves (see the discussion in Section 7).

2 Related Work

There is a large body of existing work on probing NLI models to gain insight into their reasoning abilities (Belinkov and Glass, 2019). As mentioned in Section 1, McCoy et al. (2019) find that language models fine-tuned on MNLI learn to leverage shallow heurisics to achieve exceptionally high accuracy on this dataset. Similarly, Chien and Kalita (2020) and Richardson et al. (2020) probe NLI models' performance with respect to specific syntactic and semantic phenomena (e.g. coordination, quantification, monotonicity, etc.). They find that SoTA models fine-tuned on MNLI and SNLI perform poorly on challenge examples generated to evaluate the models with respect to these phenomena, but can be easily fine-tuned to master the challenge data, while retaining their high performance on the original datasets.

Inoculation by Fine-Tuning In all three of the aforementioned papers, their respective authors utilize the method of *inoculation by fine-tuning*. Liu et al. (2019a) introduce this paradigm as a technique for differentiating between deficiencies in a model's training data and deficiencies in the model itself. Inoculation by fine-tuning assumes that there is an *original* dataset (divided into train and test splits) and a smaller *challenge* dataset (also divided into train and test splits), and that the model's performance on the challenge dataset is significantly lower than on the original dataset. The idea is to fine-tune the model on the challenge dataset until validation performance on the *original* test set has not improved for five epochs, then measure the newly fine-tuned (*inoculated*) model on the challenge test set. If the inoculated model maintains its performance on the original test set and performs (nearly) as well on the challenge test set, this suggests that the model's poor performance on the challenge data was due to flaws (e.g. a lack of diversity) in the original training data. Conversely, if the model's performance on the challenge test set remains significantly worse than on the original data after inoculation, this suggests that its poor performance on the challenge data is due to a deficiency in the model itself.

This paper probes various NLI models' logical reasoning abilities—in particular with respect to external negation—using automatically-generated challenge data along with the inoculation by fine-tuning paradigm. Unlike the closely-related notion of *adversarial attacks*, which seek to perturb input examples without altering their class labels, the external negation prefixes used to generate challenge examples in Experiments 1-3 (Sections 4, 5, 6) do alter the original examples' class labels, albeit in a predictable manner. This is similar to the challenge data that Niven and Kao (2019) construct from the Argument Reasoning Comprehension Task (Habernal et al., 2018); these authors find that BERT *cannot* be inoculated against such challenge data, and conclude that transformer models' inability to ground text to real-world concepts presents an insurmountable barrier to their logical-reasoning abilities.

Probing LMs with Negation In a similar vein, Naik et al. (2018) conduct "stress tests" on NLI models by concatenating logical distractor strings (e.g. *"and false is not true"*) to the input examples,

[1]All code used in these experiments is available here: https://github.com/mjs227/AdversarialNLI

and find that such distractors drastically reduce SoTA NLI models' performance on these tasks. While these authors investigate NLI models' performance with respect to logical reasoning, their experiments regarding negation are limited to negation items appearing in these distractor terms, rather than negating the original hypothesis sentence itself. On the other hand, Hossain et al. (2020) probe NLI models (and datasets) by negating the original premise and/or hypothesis sentence(s) using an automatic dependency parser; these automatically-generated challenge examples are then checked for accuracy and re-assigned class labels by human annotators. These authors find that models fine-tuned on the original NLI datasets perform poorly on development sets consisting of these negation-augmented examples, and that while fine-tuning on the challenge data improves performance on negation-augmented test splits derived from SNLI, fine-tuning does *not* significantly increase model accuracy on MNLI-derived examples. Note that, unlike the present work, Hossain et al. (2020) do not study repeated/embedded negation or double-negation cancellation.

Yuan et al. (2023) examine pretrained language models' (PLMs) *deductive* reasoning abilities via cloze tests. These authors find that PLMs are unable to fully generalize rules of logical deduction to arbitrary contexts. Furthermore, they observe that these models struggle to differentiate between positive statements and their negated counterparts, in line with a wide body of recent literature suggesting that transformers have difficulty processing and comprehending negation (e.g. Niven and Kao, 2019; Naik et al., 2018; Yuan et al., 2023; Laverghetta Jr. et al., 2021; Rogers et al., 2020; Ettinger, 2020; Laverghetta Jr. and Licato, 2022; Kassner and Schütze, 2020). Of particular interest to this work, they find that while inoculating PLMs for deductive reasoning tasks improves performance, it results in catastrophic forgetting of previous knowledge. Likewise, in Sections 5 and 6 of this paper, we find that inoculating pretrained NLI models against challenge data augmented with external negation prefixes causes catastrophic forgetting of prior knowledge of their similarity to related prefixes.

Jang et al. (2022) evaluate the consistency of language models across various axes. Of particular interest to the current discussion is their analysis of *negational consistency*: the degree to which a given language model's predictions differ between texts having opposite meanings. These authors find that negational consistency remains low across a variety of models and tasks—in particular, RoBERTa (Liu et al., 2019b) and BART (Lewis et al., 2020) exhibit low negational consistency on the SNLI dataset.

In an experiment highly related to the present work, Laverghetta Jr. and Licato (2022) probe NLI models' performance with respect to negation, and find that the models struggle to contend with certain types of negation more so than others. In line with the results we observe in Section 4, they find that the models have difficulty inoculating against those problematic negation categories. Unlike the experiments in this paper, Laverghetta Jr. and Licato (2022) do not construct challenge examples involving negation, but rather use examples drawn from NLI datasets that already contain negation.

Contributions Unique to this work is the evaluation of transformers' ability to learn the law of the excluded middle (LEM) and our finding that, while many cannot learn this pattern, a few transformer NLI models are in fact able to inductively learn LEM for a single external negation prefix. Additionally, the results of Experiments 2 and 3 (Sections 5 and 6), extend Yuan et al.'s (2023) results (regarding catastrophic forgetting resulting from inoculation in the context of deductive reasoning tasks) to double negation-cancellation in the setting of NLI tasks. Finally, Theorem 1 (see Section 3) is the first known proof that there exists (at least, in principle) an encoder transformer capable of modeling LEM for arbitrary-length sequences of any combination of external negation prefixes with respect to any NLI dataset. This theorem sheds further light on evidence in the literature (Niven and Kao, 2019; Naik et al., 2018; Yuan et al., 2023; Laverghetta Jr. et al., 2021; Rogers et al., 2020; Ettinger, 2020; Laverghetta Jr. and Licato, 2022; Kassner and Schütze, 2020, etc.) indicating that transformers struggle to model negation, suggesting that this observed failure is not due to an inherent flaw in transformer architectures themselves, but instead may be due to deficiencies in their training procedure and/or the structure of their input data (see the discussion in Section 7).

3 Can Transformers Model LEM?

Before evaluating NLI models' ability to inductively *learn* the law of the excluded middle (LEM), we first establish whether—learnability aside—it is theoretically possible for transformer architectures to model LEM at all: Theorem 1 proves that (encoder) transformer architectures are in fact capable of modeling LEM (at least, with respect to NLI tasks) for arbitrary-length sequences of any combination of external negation prefixes. Note that the NLI datasets, transformer models (with the exception of BART), and set of external negation prefixes used in Experiments 1-3 satisfy the assumptions of Theorem 1.

Theorem 1. *Let $D = \{(P_i, H_i, L_i)\}_{i \in I}$ be a finite-cardinality NLI dataset, and for any NLI model M, let $Acc(M, D)$ denote the classification accuracy of M on D. Let Σ' be a finite alphabet such that $D \subset (\Sigma')^* \times (\Sigma')^* \times \Lambda$ (where $\Lambda = \{\mathcal{E}, \mathcal{N}, \mathcal{C}\}$ denotes the set of labels). Let $N \subset (\Sigma')^*$ be any finite-cardinality set of external-negation prefixes such that no prefix is a substring of one or more other prefixes*[2].

Then there exists an alphabet $\Sigma \supset \Sigma'$ and an injective $f\colon (\Sigma')^* \to \Sigma^*$ such that for any fixed (finite) $w > max_{i \in I}|P_i H_i|$ and any fixed-precision transformer encoder (with an NLI classification head) T, there exists a fixed-precision transformer encoder T' such that T' matches the accuracy of T on D and on any dataset D' formed by prefixing any $\eta \in N^*$ to each hypothesis sentence in D[3].

Proof. Appendix A. □

However, the proof of Theorem 1 relies on a function f that re-structures the input data; the transformer NLI models evaluated in Experiments 1-3 (Sections 4, 5, 6) are obviously not equipped with such a function, and the ability of transformers to model LEM with respect to unstructured plain text is not established in Theorem 1. Furthermore, Theorem 1 merely states that there exists an encoder transformer capable of modeling LEM for external negation with respect to NLI tasks, and makes no claim regarding its architectural configuration (i.e. layer size, floating-point precision, etc.). It is unclear whether the transformer models evaluated in the present work have the specific architecture required to accomplish this task.

Critically, the proof of Theorem 1 does not make any claims regarding the (inductive) learnability of LEM; while it is theoretically possible to *model* LEM with an encoder transformer, these language models' ability to *inductively learn* LEM remains uncertain from the conclusions of Theorem 1 alone. In the following section (Experiment 1), we observe experimental evidence demonstrating that some transformer NLI models (in particular, RoBERTa) are able to learn LEM for a single external negation prefix.

4 Experiment 1

Experiment 1 probes six different transformer NLI models' ability to inductively learn the law of the excluded middle (LEM) with respect to external negation. The DeBERTa (He et al., 2020) model, denoted $DeBERTa_S$[4], is DeBERTa-large fine-tuned on SNLI. The first BART model, denoted $BART_M$[5], is BART-large fine-tuned on MNLI, while the second, $BART_{SMFA}$[6], is BART-large fine-tuned on MNLI, SNLI, FEVER (Thorne et al., 2018), and ANLI (Nie et al., 2020). The first RoBERTa model, $RoBERTa_M$[7], is RoBERTa-large fine-tuned on MNLI, and the second, $RoBERTa_S$[8], is RoBERTa-large fine-tuned on SNLI, while the third, $RoBERTa_{SMFA}$[9], is RoBERTa-large fine-tuned on SNLI, MNLI, FEVER, and ANLI.

4.1 Experimental Setup

For each $1 \leq n \leq 5$ and each NLI dataset $D \in \{\text{MNLI}, \text{SNLI}\}$, let $D^{\leq n}$ denote the *depth-$\leq n$ challenge set* (consisting of train and development splits). $D^{\leq n}$ is generated from examples randomly drawn from the original dataset's train splits: each MNLI$^{\leq n}$ consists of 4,906 entailment, neutral, and contradiction examples (14,718 total; 9,813 train/4,905 development), and each SNLI$^{\leq n}$ consists of 14,997 examples (4,999 per class; 9,999 train/4,998 development).

Challenge Data Generation For each $1 \leq k \leq n$, $1/n^{th}$ of the examples in each class in $D^{\leq n}$ are depth-k negated by prepending the *trigger* prefix T_{NT} = *"it is not true that"* to the original hypothesis sentence k times (i.e. by converting (P, H) to $(P, (T_{NT})^k H)$; see Table 5 in the appendix for

[2]Formally: for all $\eta \in N$, $\eta', \eta'' \in (N - \{\eta\})^*$, there does not exist i, j such that $\eta = \eta'_{i:} \mid\mid \eta''_{:j}$

[3]Formally: $Acc(T', f(D)) = Acc(T, D)$, and for any $\eta \in N^*$ such that $max_{i \in I}|P_i \eta H_i| \leq w$: $Acc(T', \{f(P_i \eta H_i)\}_{i \in I}) = Acc(T, D)$

[4]https://huggingface.co/pepa/deberta-v3-large-snli

[5]https://huggingface.co/facebook/bart-large-mnli

[6]https://huggingface.co/ynie/bart-large-snli_mnli_fever_anli_R1_R2_R3-nli

[7]https://huggingface.co/roberta-large-mnli

[8]https://huggingface.co/pepa/roberta-large-snli

[9]https://huggingface.co/ynie/roberta-large-snli_mnli_fever_anli_R1_R2_R3-nli

examples). For example, in $D^{\leq 5}$, $1/5^{th}$ of the examples in each class are depth-5 negated, $1/5^{th}$ are depth-4 negated, $1/5^{th}$ are depth-3 negated, etc. One may object that concatenating T_{NT} five times (for example) in front of the original hypothesis does not a result in a particularly natural sentence, and that a model is highly unlikely to encounter such a sentence in real-world text data. Regardless of its naturality, however, this pattern is fairly trivial (for a human) to learn: given a challenge example $(P, (T_{NT})^k H)$—derived from an original example (P, H)—simply count the number k of occurrences of T_{NT}. The class label remains the same if k is even, and *contradiction* flips to *entailment* (and vice-versa; *neutral* examples do not change their class label) if n is odd (see Appendix B.1 for a detailed discussion).

Finally, for all $m > 1$ and each NLI dataset $D \in \{\text{MNLI}, \text{SNLI}\}$, let D^m_{NT} denote the *depth-m test set*. The procedure for generating D^m_{NT} is nearly identical to that of $D^{\leq m}$ (D^m_{NT} is the size of the development split of $D^{\leq m}$), with the exception that D^m_{NT} consists *only* of depth-m externally-negated examples.

Note that the two datasets (MNLI and SNLI) contain many examples that are not complete sentences—but rather sentence fragments—in which case the external negation prefix T_{NT} = *"it is not true that"* is grammatically nonsensical. To account for this, the pool of possible examples to be included into the challenge datasets consists only of those in which the hypothesis H is a complete sentence. If the first word in H is (part of) a named entity (as determined by SpaCy's *EntityRecognizer*[10] named entitiy recognition pipeline), then the augmented (i.e. challenge) hypothesis is set to $(T_{NT})^n H$. If the first word in H does *not* belong to a named entity, then the augmented hypothesis is $(T_{NT})^n H_0$, where H_0 is formed from H by lower-casing the first character. This is to control for potential confounding factors due to irregular capitalization.

Inoculation and Evaluation For all $1 \leq n \leq 5$, each NLI model was inoculated against the challenge set(s) $D^{\leq n}$. Following the paradigm of inoculation by fine-tuning, the models were fine-tuned on the train split of $D^{\leq n}$, and validated at each epoch on the *original* NLI dataset's development split, with early-stopping if validation performance does not improve after five epochs. Once inoculated on the depth-$\leq n$ external negation data ($D^{\leq n}$), the models were evaluated on D^m_{NT} for multiple values of $m > n$. This is to measure the degree to which the models are able to generalize LEM beyond the number of external negation prefixes seen during inoculation.

Each model was evaluated and inoculated on the challenge dataset(s) generated from the dataset(s) that the model was originally fine-tuned on: $BART_M$ and $RoBERTa_M$ were evaluated on MNLI-derived examples, $RoBERTa_S$ and $DeBERTa_S$ on SNLI-derived examples, and $BART_{SMFA}$ and $RoBERTa_{SMFA}$ on examples derived from both datasets. All models were fine-tuned with a batch size of 64 at a learning rate of 10^{-5} using the Adam (Kingma and Ba, 2014) optimizer.

4.2 Results and Discussion

For the sake of brevity, model original/challenge development set accuracies pre- and post-inoculation are located in Appendix B.2. Most models were able to inoculate against the depth-$\leq n$ external negation data for all $1 \leq n \leq 5$; they retain their high-performing accuracy on the original development sets, and perform as well (or nearly so) on the challenge development sets after inoculation. The notable exceptions were $BART_M$ and $BART_{SMFA}$, which struggled to inoculate for $n \in \{1, 4, 5\}$ and $n \in \{3, 4\}$, respectively—recall that BART is the only model architecture evaluated in this experiment that does not satisfy the assumptions of Theorem 1.

In spite of their ability to inoculate against depth-$\leq n$ challenge data, the models struggled to generalize this knowledge to depth-m negation for values of $m > n$. Table 1 reports average model accuracy (individual model accuracies are located in Appendix B.3) on depth-m>n external negation after depth-$\leq n$ inoculation for $1 \leq n \leq 3$, $2 \leq m \leq 6$. A clear pattern emerges in this table: before any inoculation, we observe high model accuracy ($\sim$80%) on the depth-m negtion data for even values of m, and near-random-chance accuracy ($\sim$34%) for odd values of m. This indicates that, before inoculation, the models were essentially entirely ignoring the external negation prefixes and treating them as distractors; depth-m negation does not alter the class label for even values of m, and so a model treating the prefix as a distractor will retain high accuracy on those examples, purely by chance. To reiterate: these mod-

[10] https://spacy.io/api/entityrecognizer

Depth-m test	No inoc.	Depth-1 inoc.	Depth-$\leq$2 inoc.	Depth-$\leq$3 inoc.
2	0.72	0.39	—	—
3	0.36	0.86	0.32	—
4	0.84	0.39	0.95	0.35
5	0.32	0.82	0.32	0.91
6	0.86	0.43	0.95	0.35

Table 1: Average accuracy across all models on depth-(m>n) external negation (D^m_{NT}) after depth-$\leq n$ inoculation ($n \in \{1, 2, 3\}$) on $D^{\leq n}$. For the sake of brevity, individual results for each model are located in Appendix B.3; most individual model accuracies do not substantially deviate from the mean values in this table.

Model	Depth-m test	No inoc.	Depth-$\leq$4 inoc.
$BART_M$	5	0.33	0.34
$RoBERTa_M$	5	0.32	0.34
$DeBERTa_S$	5	0.30	0.33
$RoBERTa_S$	5	**0.34**	**0.89**
$BART_{SMFA}$	5	0.30	0.32
$RoBERTa_{SMFA}$	5	**0.32**	**0.79**
$BART_M$	6	0.86	0.93
$RoBERTa_M$	6	0.89	0.95
$DeBERTa_S$	6	0.88	0.95
$RoBERTa_S$	6	**0.83**	**0.93**
$BART_{SMFA}$	6	0.86	0.93
$RoBERTa_{SMFA}$	6	**0.84**	**0.95**

Table 2: Accuracy for all models on depth-m external negation after depth-$\leq$4 inoculation ($m \in \{5, 6\}$).

Model	Depth-m test	No inoc.	Depth-$\leq$5 inoc.
$BART_M$	6	0.86	0.34
$RoBERTa_M$	6	**0.89**	**0.91**
$DeBERTa_S$	6	0.88	0.31
$RoBERTa_S$	6	**0.83**	**0.94**
$BART_{SMFA}$	6	0.86	0.30
$RoBERTa_{SMFA}$	6	**0.84**	**0.95**
$BART_M$	7	0.32	0.93
$RoBERTa_M$	7	**0.32**	**0.96**
$DeBERTa_S$	7	0.28	0.95
$RoBERTa_S$	7	**0.36**	**0.94**
$BART_{SMFA}$	7	0.29	0.92
$RoBERTa_{SMFA}$	7	**0.31**	**0.95**

Table 3: Accuracy for all models on depth-m external negation after depth-$\leq$5 inoculation ($m \in \{6, 7\}$).

els—ostensibly fine-tuned on a logical-reasoning task—*have learned to entirely ignore external negation* when predicting inferential relations.

Furthermore, when inoculated against depth-1 external negation, the pattern reverses: we note near-random-chance accuracy for *even* values of m, and high accuracy for *odd* values of m. After depth-1 inoculation, the models have learned to treat *any* depth-m external negation prefix as equivalent to a depth-1 (i.e. single) prefix.

Interestingly, after depth-$\leq$2 inoculation, the models revert to the original pattern of high accuracy for even values of m, and poor performance for odd values. Despite training on both depth-1 and depth-2 external negation, the models merely memorize the effect of depth-1 negation on class labels, and do not generalize to odd values of $m > 1$. A similar pattern emerges after depth-$\leq$3 inoculation: after fine-tuning on depth-1, depth-2, and depth-3 external negation, the models memorize the effect (or lack thereof) of depth-2 negation on class labels, and do not generalize to even values of $m > 2$.

However, Table 2 indicates that, after depth-$\leq$4 inoculation, two of the RoBERTa models ($RoBERTa_S$ and $RoBERTa_{SMFA}$) do in fact inductively learn to repeatedly cancel double negation for values of $m > 4$. After depth-$\leq$5 inoculation, $RoBERTa_M$ also learns the desired pattern (see Table 3); all three RoBERTa models have inductively learned LEM for arbitrary values of m.

Given all six models' difficulty with inoculation against depth-m external negation (for arbitrary values of m), it is reasonable to question the RoBERTa models' ability to generalize the negation-cancellation patterns that they have learned after depth-$\leq$5 inoculation to external negation strings beyond the trigger T_{NT} = *"it is not true that"* that they saw during inoculation. The following experiment (Section 5) evaluates the three RoBERTa models' ability to repeatedly cancel double negation with respect to the prefix *"it is $\overline{\text{false}}$ that"*, after inoculation against $D^{\leq 5}$ (i.e. depth-$\leq$5 *"it is not true that"* prefixes).

5 Experiment 2

This experiment restricts its analysis to the three RoBERTa models, as they were the only models of the six evaluated in Experiment 1 (Section 4) that were able to fully generalize depth-m negation-cancellation to arbitrary values of $m > 5$.

5.1 Experimental Setup

For all $m \geq 1$ and each NLI dataset $D \in \{\text{MNLI}, \text{SNLI}\}$, let D_F^m denote the depth-m challenge test set. Each D_F^m was created in an identical manner to the depth-m challenge test sets D_{NT}^m defined in Section 4.1 above: D_F^m consists only of examples (drawn from the dataset's original development split) modified to have depth-m externally-negated hypothesis sentences with an equal number of examples per class label (and $|D_F^m| = |D_{NT}^m|$).

However, in place of the trigger T_{NT} = *"it is not true that"* used to construct D_{NT}^m, in this experiment D_F^m was generated using the trigger T_F = *"it is false that"*. These two triggers are effectively semantically equivalent; the phrase *"not true"* has simply been replaced with the (effectively) synonymous *"false"*. Assuming that the models have truly learned the law of the excluded middle (LEM), we should expect to see similar performance on D_F^m to that of D_{NT}^m.

After inoculation on the depth-$\leq$5 T_{NT} external negation data, each of the three RoBERTa models ($RoBERTa_S$, $RoBERTa_M$, $RoBERTa_{SMFA}$) was evaluated on D_F^m for all $1 \leq m \leq 8$. As in the procedure for Experiment 1 (see Section 4.1), each model was evaluated on the challenge dataset(s) generated from the dataset(s) that the model was originally fine-tuned on.

5.2 Results and Discussion

Figure 1 shows the results of this experiment: $RoBERTa_S$ failed to generalize LEM from T_{NT} to T_F for values of $m > 2$, while $RoBERTa_M$ and $RoBERTa_{SMFA}$ experience precipitous decreases in accuracy at $m = 5$ (and erratic accuracy thereafter). While these models are able to generalize external negation-cancellation to arbitrary-length repeated T_{NT} prefixes, they clearly cannot extend this pattern to near-synonymous prefixes.

We may object that the models have failed to learn the pattern for T_F because they did not see it during inoculation. This objection may be valid, but belies the critical point: *these models have failed to generalize LEM* from T_{NT} to T_F. While

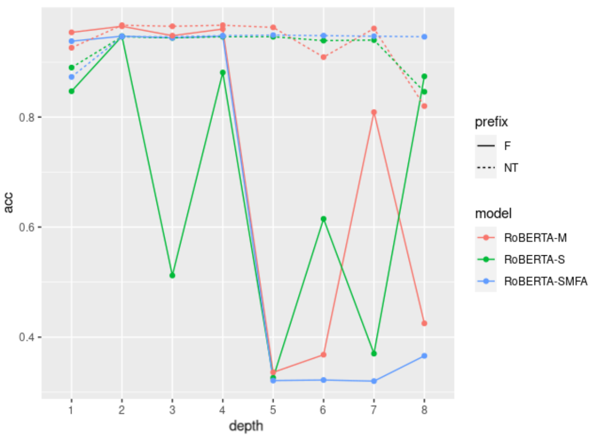

Figure 1: Accuracy for the depth-$\leq$5 T_{NT}-inoculated RoBERTa models on depth-m externally-negated examples with T_{NT} (dashed) and T_F (solid).

the models very well may learn to cancel external negation prefixes after fine-tuning on all possible sequences of this type (see the discussion in Section 7), at that point they are not learning—but rather memorizing—the pattern.

Given the conclusions of Theorem 1 and the RoBERTa models' ability to generalize double-negation cancellation for T_{NT} as observed in Experiment 1 (Section 4), the results of Experiment 2 beg the question as to *why* the RoBERTa models cannot fully generalize LEM from T_{NT} to T_F. The following experiment (Section 6) examines the embeddings generated by the RoBERTa models pre- and post-inoculation, shedding light on the root of their failure to learn to generalize LEM to arbitrary prefixes.

6 Experiment 3

As in Experiment 2 (Section 5), this experiment restricts its analysis to the three RoBERTa models.

6.1 Experimental Setup

As mentioned above, this experiment probes the embeddings that these models generate before and after depth-$\leq$5 T_{NT} inoculation. The experiment proceeds as follows: for each dataset $D \in \{\text{MNLI}, \text{SNLI}\}$, take a subset D' of the original development set (D' contains ~50-100 examples of each class, depending on the size of the dataset). For each $1 \leq m \leq 8$ and each $(P_i, H_i) \in D'$, compute the cosine similarity between the (mean-pooled) embeddings of $(T_{NT})^m H_i$ and $(T_F)^m H_i$.

For even values of m, compute the cosine similarity between $(T_{NT})^m H_i$ and $(T_F)^2 H_i$;

$(T_{NT})^2 H_i$ and $(T_F)^m H_i$; $(T_{NT})^m H_i$ and $(T_{NT})^2 H_i$; $(T_F)^m H_i$ and $(T_F)^2 H_i$; $(T_{NT})^m H_i$ and H_i; and $(T_F)^m H_i$ and H_i. For odd m, compute the similarity between $(T_{NT})^m H_i$ and $(T_{NT})^1 H_i$; $(T_F)^m H_i$ and $(T_F)^1 H_i$; $(T_{NT})^m H_i$ and $(T_F)^1 H_i$; and $(T_F)^m H_i$ and $(T_{NT})^1 H_i$.

As in Experiments 1 and 2 (Sections 4 and 5, respectively), each model was evaluated using the challenge dataset(s) generated from the dataset(s) that the model was originally fine-tuned on.

6.2 Results and Discussion

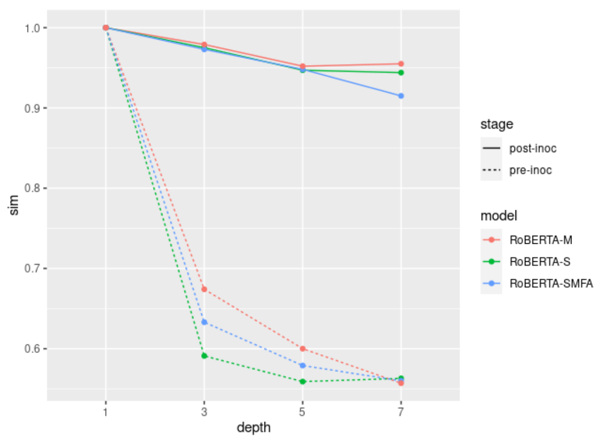

Figure 2: Mean cosine similarity between $(T_{NT})^n H_i$ and $(T_{NT})^1 H_i$ for the three RoBERTa models before (dashed) and after (solid) depth-$\leq$5 T_{NT} inoculation.

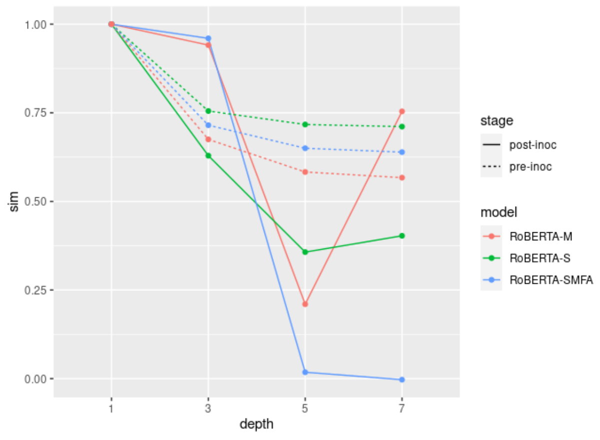

Figure 3: Mean cosine similarity between $(T_F)^n H_i$ and $(T_F)^1 H_i$ for the three RoBERTa models before (dashed) and after (solid) depth-$\leq$5 T_{NT} inoculation.

For the sake of brevity, Appendix C reports the majority of the results of this experiment.

We observe that depth-$\leq$5 inoculation drastically increases the similarity between $(T_{NT})^m H_i$ and $(T_{NT})^1 H_i$ for all three models for odd values of m (Figure 2), but *decreases* the similarity between $(T_F)^m H_i$ and $(T_F)^1 H_i$ for $m \geq 5$ (Figure 3)—recall that for odd m, $(T_{NT})^m H_i$ / $(T_F)^m H_i$ should be synonymous with $(T_{NT})^1 H_i$ / $(T_F)^1 H_i$. The results are analogous for even values of m (see Figures 4-7 in the appendix).

Model	Before	After
$RoBERTa_M$	0.996	0.268
$RoBERTa_S$	0.996	0.712
$RoBERTa_{SMFA}$	0.996	0.646

Table 4: Cosine similarity between the RoBERTa models' (mean-pooled) embeddings of the strings *"false"* and *"not true"* before and after depth-$\leq$5 inoculation.

Additionally, as m increases, mean cosine similarity decreases between $(T_F)^m H_i$ and $(T_{NT})^2 H_i$, and $(T_F)^m H_i$ and $(T_{NT})^1 H_i$ (see Figures 8 and 10 in the appendix, respectively). We also observe decreases in cosine similarity between $(T_F)^m H_i$ and $(T_{NT})^m H_i$ for even and odd $m > 4$ (see Figure 12 in the appendix).

These results indicate that the inoculation procedure conducted in Experiment 1 (Section 4) has lead to catastrophic forgetting. In particular, it seems that learning to cancel double negation for T_{NT} has drastically altered the models' encodings of the string *"not true"*, pulling its representation in the embedding space away from those of similar phrases such as *"false"*. This conjecture is supported by Table 4: we observe that—before inoculation—the models' representations of the strings *"not true"* and *"false"* are nearly identical. However, after depth-$\leq$5 T_{NT} inoculation, the models' representations of the two strings are substantially further apart in the embedding space.

Furthermore, the results of this experiment indicate that the models have not learned the linguistic function of negation during pre-training or original fine-tuning on the MNLI and SNLI datasets, analogous to the findings of Yuan et al. (2023) with respect to deductive reasoning tasks. Aside from the results in Table 1 indicating that these NLI models simply treat external negation prefixes as distractors (before inoculation), note that if the models already understood the logical function of prefixes such as T_{NT}, then further refining the models' knowledge of the function of that prefix (i.e. fine-tuning on the depth-$\leq$5 T_{NT} data) should not significantly alter its representation in the embedding space relative to highly similar prefixes such as T_F, contrary to what we observe in Table 4.

7 Discussion

The results of Experiments 1-3 (Sections 4, 5, 6) raise the question as to *why* these models are unable to inductively learn the law of the excluded middle (LEM)—especially in light of Theorem 1, which states that (in theory) transformers are able to model LEM with respect to NLI tasks. A reasonable explanation for this seemingly paradoxical state of affairs can be found within the conclusions of Theorem 1 itself.

Note that the proof of Theorem 1 relies on a function f that re-structures the input data (as mentioned in Section 3); it is possible that the structure (or lack thereof) of purely textual data may be insufficient for transformers to inductively learn to model LEM.

Additionally, recall that the proof of Theorem 1 does *not* establish the (inductive) *learnability* of LEM; it may be the case that the specific parameter values required to model the role of (external) negation in the context of NLI tasks cannot be reached by training on any NLI dataset using gradient descent or any other currently known training procedures. It may also be the case that the function of (external) negation is in fact learnable, but only via the brute-force approach of training these models on multiple-depth external negation for every such prefix. In other words, (encoder) transformers may not be capable of *inductively* learning LEM—at least not with standard training procedures.

Hosseini et al. (2021) and Asai and Hajishirzi (2020) propose training procedures designed to enhance language models' ability to learn the role of negation, which may provide fruitful avenues for improving transformer NLI models' performance on the tasks laid out in Experiments 1-3 (Sections 4, 5, 6). Hosseini et al. (2021) introduce *unlikelihood with reference* training for masked language models, which penalizes models for predicting unlikely tokens in negated contexts—for example, a model would be penalized for predicting *fly* in the context *"birds cannot [MASK]"*. After unlikelihood with reference training, the authors record marginal improvement ($\sim$1-2%) for BERT on negation-augmented SNLI and MNLI datasets (see Hossain et al., 2020).

Asai and Hajishirzi (2020) use logic-based regularization and data augmentation to improve language models' *transitive* and *symmetric* consistency (c.f. Jang et al., 2022)—in particular, negation is subsumed under their notion of symmetric consistency. Using this approach, the authors record marked improvement over the SoTA on a variety of question-answering tasks, although they do not evaluate this regularization method on any NLI datasets.

However, Hosseini et al. (2021) and Asai and Hajishirzi (2020) do not explicitly study the efficacy of their respective training methods with respect to double-negation cancellation. Therefore, it is unclear whether the improvements obtained by their approaches would translate to a task such as LEM, and we leave an evaluation thereof to future work.

8 Conclusion

The results of Experiments 1-3 (Sections 4, 5, 6) demonstrate that near-SoTA transformer NLI models struggle to inductively learn the law of the excluded middle (LEM). Furthermore, the results of Experiment 1 (Section 4) strongly suggest that all six NLI models studied in this work learned to treat the external negation prefix *"it is not true that"* as a distractor when initially fine-tuned on the NLI dataset(s) (see Table 1). Experiment 1 also suggests that DeBERTa and BART models are incapable of learning to inductively generalize LEM, despite extensive fine-tuning.

These findings lend further support to a large body of existing evidence (e.g. Niven and Kao, 2019; Naik et al., 2018; Yuan et al., 2023; Laverghetta Jr. et al., 2021; Rogers et al., 2020; Ettinger, 2020; Laverghetta Jr. and Licato, 2022; Kassner and Schütze, 2020) indicating that transformers are unable to model the meaning of negation. Unique to this work is our finding that certain encoder transformers (in particular, RoBERTa) can learn LEM for a single external negation prefix.

While the three RoBERTa models did manage to grasp the function of the prefix *"it is not true that"*, the process of learning this behavior resulted in catastrophic forgetting, entirely inhibiting their generalization of this pattern to the highly similar prefix *"it is false that"* (see Sections 5 and 6).

Theorem 1 proves that encoder transformers are—in principle—capable of modeling LEM for arbitrary-length sequences of any combination of external negation prefixes with respect to any NLI dataset. This suggests that these models' inability to inductively learn LEM might not be a consequence of their transformer architectures, but rather may result from the (lack of) structure of their input data and/or the procedure used to train them.

9 Limitations

While Experiments 1-3 (Sections 4, 5, 6) probe a variety of encoder and encoder-decoder transformers, they do not consider decoder-only models such as LLaMa-2 (Touvron et al., 2023) or GPT-3 (Brown et al., 2020); evaluation of decoder transformers is left to future work. Additionally, these experiments only utilize MNLI and SNLI for challenge data generation and evaluation, although both datasets have been shown to consist of non-representative data and contain annotation artifacts that permit models to achieve high performance by leveraging shallow heuristics (McCoy et al., 2019; Richardson et al., 2020). However, the use of more challenging NLI datasets such as ANLI was precluded by all six models' (including those fine-tuned on ANLI) already-poor performance on the original ANLI test set prior to any inoculation.

The main limitation regarding the challenge datasets themselves is the fact that they we generated using only two external negation prefix triggers: *"it is true that"* and *"it is false that"*. While this suffices to demonstrate the models' inability to inductively learn the law of the excluded middle (LEM) and/or generalize this knowledge to similar prefixes, future work should incorporate a wider variety of negation triggers in similar experimental settings.

Note that Theorem 1 applies only to *encoder* transformers, as the proof is formulated using a variant of first-order logic (FOC[+;MOD]; Immerman, 2012) that has only been shown to be an upper-/lower-bound for fixed-precision encoder transformers (Chiang et al., 2023). Additionally, the proof of Theorem 1 requires a fixed input length w; while the input sequence length of all "real-world" transformers is practically bounded by the quadratic growth rate of their self-attention mechanism (Beltagy et al., 2020), this assumption of a fixed input size still presents a limitation to the scope of the theorem.

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A Proof of Theorem 1

A.1 FOC[+;MOD]

Chiang et al. (2023) prove that FOC[+;MOD] (a variant of first-order logic defined over strings over a finite alphabet Σ; see Immerman, 2012) is both an upper bound for fixed-precision transformer encoders and a lower bound for arbitrary-precision encoder transformer encoders, in the sense that every language that is recognizable by a fixed-precision encoder transformer (binary) classifier is definable by a sentence of FOC[+;MOD] (Chiang et al., 2023, Theorem 2), and every language defined by a sentence of FOC[+;MOD] is recognizable by an (arbitrary-precision) encoder transformer (binary) classifier (Chiang et al., 2023, Theorem 5). Given an FOC[+;MOD] formula ϕ, the language defined by ϕ is the set of all strings $\sigma \in \Sigma^*$ such that ϕ holds with respect to σ.

The syntax of FOC[+;MOD] consists of two sorts:

- *Positions*: (positive) integer variables p that range over positions in strings σ.
- *Counts*: variables x ranging over the rational numbers, and terms $c_0 + c_1x_1 + \cdots + c_nx_n$, where each c_i is a (constant) rational number and each x_i is a count variable.

Formulas of FOC[+;MOD] are defined as one of:

- $\top$ (true) or $\bot$ (false).
- $Q_a(p)$, where $a \in \Sigma$, and $Q_a(p) := \sigma_p = a$
- $MOD_b^a(p)$, where $a \geq 0$, $b > 0$, and p is a position variable; $MOD_b^a(p) := p \equiv_b a$
- $\phi \wedge \psi$, $\phi \vee \psi$, or $\neg\psi$, where ϕ and ψ are formulas.[11]
- $x_1 = x_2$ or $x_1 < x_2$, where x_1, x_2 are in the sort of counts.[12]
- $\exists x.\phi$ or $\forall x.\phi$, where x is a count variable and ϕ is a formula.
- $\exists^{=x} p.\phi$, where x is a count variable, p is a position variable ($\exists^{=x} p.\phi$ binds p but leaves x free), and ϕ is a formula; $\exists^{=x} p.\phi$ holds if and only if ϕ is true for exactly x values of p.

In particular, note that FOC[+;MOD] does *not* permit arithmetic operations (addition or multiplication) or comparisons ($=$, $<$) of position variables, only of count variables. This is the primary motivation for much of the machinery introduced in the proof of Theorem 1 (Appendix A.3).

A.2 Notation

We now introduce additional notation employed in the proof of Theorem 1:

- $\sigma \mid\mid \sigma'$: denotes the concatenation of the strings σ and σ'. Note that when convenient (and unambiguous), we omit the operator and write $\sigma\sigma'$ to denote $\sigma \mid\mid \sigma'$.
- $\overset{n}{\underset{i=k}{\mid\mid}} (\dots)$: denotes iterated string concatenation.
- $|\sigma|$: unless otherwise specified, denotes the length of the string σ.
- σ_i: denotes the i^{th} character of the string σ.
- $\Sigma^* = \bigcup_{i=1}^{\infty} \Sigma^i$: denotes the set of all *non-empty* strings over the alphabet Σ. Note that unless otherwise specified, we slightly abuse

[11]We can derive $\phi \rightarrow \psi$ and $\phi \leftrightarrow \psi$ as $\psi \vee \neg\phi$ and $\phi \rightarrow \psi \wedge \psi \rightarrow \phi$, respectively.

[12]We can derive $x_1 \leq x_2$ as $x_1 = x_2 \vee x_1 < x_2$, $x_1 > x_2$ as $x_2 < x_1$, $x_1 \geq x_2$ as $x_2 \leq x_1$, and $x_1 \neq x_2$ as $\neg(x_1 = x_2)$.

notation and let A^* (for any $A \subseteq \Sigma^*$) denote the set of "flattened" strings of A—i.e. $A^* = \bigcup_{i=1}^{\infty} \bigcup_{a \in A^i} \{ \Vert_{k=1}^{|a|} a_k \}$ so that for all $a' \in A^*$, $a' \in \Sigma^*$.

- ϵ: denotes the empty string.
- $\sigma_{i:j} = \Vert_{k=i}^{j} \sigma_k$: denotes the substring spanning the i^{th} to j^{th} (inclusive) characters of σ; if $i = j$, then $\sigma_{i:j} = \sigma_i$.
- $\sigma_{i:}$, $\sigma_{:j}$: denote $\sigma_{i:|\sigma|}$ when $i \leq |\sigma|$ and $\sigma_{1:j}$ when $j \geq 1$, respectively. If $i > |\sigma|$, then $\sigma_{i:} = \epsilon$.
- $\sigma^n = \Vert_{i=1}^{n} \sigma$: denotes the string σ repeated n times ($\sigma^0 = \epsilon$).
- $\phi[x \Rightarrow y] = \lambda x.[\phi](y)$: denotes the formula obtained from ϕ by replacing all instances of the free variable x with the variable (or constant) y.
- $[\phi](\sigma) = \sigma \models \phi$: indicates that the formula ϕ holds for the string σ (i.e. σ belongs to the language defined by ϕ).

A.3 Proof

Let $\Lambda = \{\mathcal{E}, \mathcal{N}, \mathcal{C}\}$ denote the set of NLI labels and let Σ' denote the input alphabet of (i.e. set of tokens for) the transformer T—we assume without loss of generality that Λ and Σ' are disjoint (i.e. $\Lambda \cap \Sigma' = \emptyset$); Theorem 1 applies only to *encoder* transformers, so we need not consider the labeling approach taken by encoder-decoder or decoder-only transformers.

By Chiang et al. (2023) Theorem 2, T corresponds to the FOC[+;MOD] formula S_T defined in Equation 1. To be explicit: Chiang et al. (2023) Theorem 2 guarantees that there exists some FOC[+;MOD] formula S_T that defines the language recognized by T. For each $(P_k, H_k, L_k) \in D$, the input to S_T is the string $P_k H_k L_k$: for all $x \in \Lambda$, $[S_T](P_k H_k L_k)$ holds if and only if the transformer T assigns the label L_k to (P_k, H_k).

$$S_T := \bigwedge_{x \in \Lambda} \phi_x \leftrightarrow \exists^{=1} p. Q_x(p) \tag{1}$$

Note that we may assume the existence of $\phi_{\mathcal{E}}$, $\phi_{\mathcal{N}}$, and $\phi_{\mathcal{C}}$ (and therefore S_T) as in Equation 1 without loss of generality. Regardless of the approach that the particular transformer T takes to predicting labels, the output of T with respect to an input $\sigma \in (\Sigma')^*$ ($\mathcal{O}_T(\sigma)$) must be an element of Λ. As such, for each $x \in \Lambda$ and $\sigma \in (\Sigma')^*$, $[\phi_x](\sigma) := \mathcal{O}_T(\sigma) = x$.

Let $\Sigma = \Sigma' \cup \{\Omega\}$, where Ω is a special padding character introduced for formal reasons, and distinct from the actual padding character used by the transformer T. For any $\sigma \in (\Sigma')^*$, define $f(\sigma) \in \Sigma^*$ as follows in Equation 2 (where w is the fixed input length specified in Theorem 1).

$$f(\sigma) := \Vert_{i=1}^{|\sigma|+1} \left(\Omega^{i-1} \Vert \sigma_{i:} \Vert \Omega^{w-|\sigma|} \right) \tag{2}$$

For all (integer) count terms $1 < b \leq w$, define $MODC_b(a, x)$ (where a, x are count variables) as follows (Equation 3):

$$MODC_1(a, x) := \top \tag{3a}$$

$$MODC_b(a, x) := \bigvee_{y=-w}^{w} yb + a = x \tag{3b}$$

Note that by Chiang et al. (2023) Theorem 1, we may assume without loss of generality that each ϕ_x in Equation 1 is in normal form (for some integer $k \geq 0$), as in Equation 4.

$$\phi_x = \exists z_1 \ldots \exists z_k [\bigwedge_{i=1}^{k} \exists^{=z_i} p.(\phi_x)_i \wedge \chi] \tag{4}$$

Where each $(\phi_x)_i$ is quantifier-free and has no free count variables, and χ is quantifier-free.

Now, for each such $(\phi_x)_i$, construct $\alpha((\phi_x)_i)$ as follows: for each $a \in \Sigma'$ such that $Q_a(p)$ appears in $(\phi_x)_i$, replace $Q_a(p)$ with $Q'_a(p)$ as defined in Equation 5 (where p is a position variable in the former, and a count variable in the latter), and replace each instance of a modular predicate $MOD^x_y(p)$ with $MODC_y(x, p)$ (where again p is a position variable in the former, and a count variable in the latter).

$$Q'_a(p) := \exists^{=p} p' [Q_a(p') \wedge \bigvee_{i=1}^{w} (MOD^i_w(p') \wedge p = i)] \tag{5}$$

Lemma 1. *For any $\sigma \in (\Sigma')^*$ such that $|\sigma| \leq w$, all $a \in \Sigma'$, and all $1 \leq p \leq w$: $[Q'_a(p)](f(\sigma)) \leftrightarrow [Q_a(p)](\sigma)$*

Proof. First, assume $[Q_a(p)](\sigma)$ holds. By assumption, $\sigma_p = a$, so by construction (Equation 2), $f(\sigma)_{yp} = a$ for all $1 \le y \le p$ and $f(\sigma)_{y'p} = \Omega$ for all $y' > p$. Therefore $[Q_a(p)](\sigma) \to [Q'_a(p)](f(\sigma))$ by definition (Equation 5).

Now, assume $[Q'_a(p)](f(\sigma))$ holds. By assumption and construction (Equation 2), $f(\sigma)_{yp} = a$ for all $1 \le y \le p$, so in particular $f(\sigma)_p = a$. By construction, $f(\sigma)_{:|\sigma|} = \sigma$. This implies that $\sigma_p = a$; therefore $[Q'_a(p)](f(\sigma)) \to [Q_a(p)](\sigma)$. □

Now, for any count variables p, z and any FOC[+;MOD] formula ϕ, define $E(p, z, \phi)$ as follows (Equation 6).

$$E_1^i(p,\phi) := \bigwedge_{j=1}^{i} \phi[p \Rightarrow m_j] \wedge m_j \le w \tag{6a}$$

$$E_2^i := \bigwedge_{a=1}^{i-1} \bigwedge_{b=a+1}^{i} m_a \ne m_b \tag{6b}$$

$$E_3^{i+2}(p,\phi) := \exists m_1 \dots m_{i+2}[E_1^{i+2}(p,\phi) \wedge E_2^{i+2}] \tag{6c}$$

$$E_3^1(p,\phi) := \exists m_1 . E_1^1(p,\phi) \tag{6d}$$

$$E_3^0(p,\phi) := \top \tag{6e}$$

$$E(p,z,\phi) := \bigvee_{i=0}^{w} (E_3^i(p,\phi) \wedge z = i) \tag{6f}$$

Where $E_1^i(-,-)$, E_2^i, and $E_3^i(-,-)$ are defined for all integers $1 \le i \le w$, $2 \le i \le w$, and $0 \le i \le w$, respectively.

Now, for each $(\phi_x)_i$ in Equation 4, define $A((\phi_x)_i)$ as in Equation 7 (where z_i and p are free count variables).

$$A_1((\phi_x)_i) := E(p, z_i, \alpha((\phi_x)_i)) \tag{7a}$$

$$A_2((\phi_x)_i) := \neg\exists y[y > z_i \wedge E(p, y, \alpha((\phi_x)_i))] \tag{7b}$$

$$A((\phi_x)_i) := A_1((\phi_x)_i) \wedge A_2((\phi_x)_i) \tag{7c}$$

Lemma 2. *For any $\sigma \in (\Sigma')^*$ such that $|\sigma| \le w$, all $x \in \Lambda$, and all $(\phi_x)_i$ as in Equation 4: $[\exists z_i \exists^{=z_i} p.(\phi_x)_i](\sigma) \leftrightarrow [\exists z_i . A((\phi_x)_i)](f(\sigma))$*

Proof. First, note that $(\phi_x)_i$ is quantifier-free and has no free count variables (Chiang et al., 2023, Theorem 1); therefore $(\phi_x)_i$ consists only of positional ($Q_a(p)$) and modular ($MOD_y^x(p)$) predicates (where the only bound position variable is p) and logical operators acting on them. $A((\phi_x)_i)$ is constructed from $(\phi_x)_i$ by replacing each instance of $Q_a(p)$ and $MOD_y^x(p)$ with $Q'_a(p)$ and $MODC_y(x,p)$ (where p is a position variable in the first pair of terms, and a count variable in the second), respectively.

By Lemma 1, $[Q_a(p)](\sigma) \leftrightarrow [Q'_a(p)](f(\sigma))$ for all $1 \le p \le w$, where p is a position variable in the left-hand side of the equation and a count variable in the right-hand side. Similarly, for all p, x and all $1 \le y \le w$, $MOD_y^x(p) \leftrightarrow MODC_y(x,p)$ by construction (Equation 3), where again p is a position variable in the left-hand side of the equation and a count variable in the right-hand side.

Therefore, for all $1 \le p \le w$, $(\phi_x)_i$ holds with respect to σ if and only if $\alpha((\phi_x)_i)$ holds with respect to $f(\sigma)$.

By construction (Equation 6), $E(p, z, \phi)$ holds for any predicate ϕ with the count variable p free if and only if there are $\ge z$ unique values of p such that ϕ holds. By definition (Equation 7), $A((\phi_x)_i)$ holds if and only if there are exactly z_i values of p such that $\alpha((\phi_x)_i)$ holds. □

Now, for each ϕ_x in Equation 1, we define $A(\phi_x)$ as in Equation 8.

$$A(\phi_x) := \exists z_1 \dots \exists z_k [\bigwedge_{i=1}^{k} A((\phi_x)_i) \wedge \chi] \tag{8}$$

Lemma 3. *For all $x \in \Lambda$ and all $\sigma \in (\Sigma')^*$ such that $|\sigma| \le w$: $[\phi_x](\sigma) \leftrightarrow [A(\phi_x)](f(\sigma))$*

Proof. By Lemma 2, each $A((\phi_x)_i)$ of Equation 8 holds for $f(\sigma)$ if and only if each $(\phi_x)_i$ holds for σ. As such, for each bound count variable z_i, the set (of cardinality z_i) of count values that make $A((\phi_x)_i)$ true with respect to $f(\sigma)$ is identical to the set of position values that make $(\phi_x)_i$ true with respect to σ. The predicate χ contains no position variables (Chiang et al., 2023, Theorem 1), and is defined identically in Equation 8 as in Equation 4; therefore, χ (within $A(\phi_x)$) holds for $f(\sigma)$ if and only if χ (within ϕ_x) holds for σ. □

Now, for each external negation prefix $\eta \in N$, define $\psi_\eta(i)$ and $\psi'_\eta(i,j)$ (where i and j are count variables) as in Equation 9, where $Q'_{(-)}(-)$ is defined as in Equation 5.

$$\psi_\eta(i) := \bigwedge_{k=0}^{|\eta|-1} Q'_{\eta_k}(i+k) \tag{9a}$$

$$\psi'_\eta(i,j) := \psi_\eta(i) \wedge i + |\eta| - 1 = j \tag{9b}$$

Then define $\psi(i)$ and $\psi'(i,j)$ (where i and j are count variables) as in Equation 10.

$$\psi(i) := \bigvee_{\eta \in N} \psi_\eta(i) \tag{10a}$$

$$\psi'(i,j) := \bigvee_{\eta \in N} \psi'_\eta(i,j) \tag{10b}$$

Now define $\rho(i,j)$ (where i and j are count variables) as in Equation 11.

$$\rho_1(k,a,b,i,j) := i \leq a \leq k \wedge k \leq b \leq j \wedge \psi'(a,b) \tag{11a}$$

$$\rho(i,j) := \forall k[i \leq k \leq j \rightarrow \exists a,b.\rho_1(k,a,b,i,j)] \tag{11b}$$

Lemma 4. *For any $\sigma \in (\Sigma')^*$ such that $|\sigma| \leq w$, and all $1 \leq i < j \leq w$: $[\rho(i,j)](f(\sigma)) \leftrightarrow \sigma_{i:j} \in N^*$ (i.e. $\rho(i,j)$ holds for $f(\sigma)$ if and only if the span $i \rightarrow j$ in σ is a sequence of one or more external negation prefixes).*

Proof. We first prove the right-to-left direction: $\sigma_{i:j} \in N^* \rightarrow [\rho(i,j)](f(\sigma))$. The proof proceeds by induction. First, assume that σ is a single external negation prefix (i.e. $\sigma_{i:j} \in N$). Then by assumption and definition (Equation 9), $\psi'_{\sigma_{i:j}}(i,j)$ holds; by definition (Equation 10), this implies $\psi'(i,j)$. For all $i \leq k \leq j$, let $a = i$, $b = j$: by definition (Equation 11), $\rho_1(k,a,b,i,j)$ holds. This implies $\rho(i,j)$. This proves the base case.

Now suppose $\sigma_{i:j} = \eta \mid\mid \eta'$, with $\eta \in N^*$ and $\eta' \in N$. By the inductive hypothesis, $\rho(i, i+|\eta|-1)$ holds. By the base case above, $\rho(i+|\eta|, j)$ holds. It now remains to prove that $\rho(i, i+|\eta|-1) \wedge \rho(i+|\eta|, j) \rightarrow \rho(i,j)$. For all $1 \leq k \leq j$, if $k < i+|\eta|$, then there exist $a, b < i+|\eta|$ such that $\rho_1(k,a,b,i,j)$ (by the validity of $\rho(i, i+|\eta|-1)$), and if $k \geq i+|\eta|$, there exist $a, b \geq i+|\eta|$ such that $\rho_1(k,a,b,i,j)$ (by the validity of $\rho(i+|\eta|, j)$); therefore, $\rho(i,j)$ holds. This proves the induction step.

We now prove the right-to-left direction by contradiction: assume $\rho(i,j)$ and $\sigma_{i:j} \notin N^*$. By assumption, there exists $\eta \in N^* \cup \{\epsilon\}$ such that η is a substring of $\sigma_{i:j}$. For all $i \leq k \leq j$ such that σ_k is not contained within η: $\neg\exists a,b.\rho_1(k,a,b,i,j)$, by the assumption that external negation prefixes do not overlap (see Theorem 1). Therefore, $\rho(i,j)$ does not hold—this is a contradiction. ☐

Now define $\rho'(i,j)$ as in Equation 12.

$$\rho'_1(a,b,i,j) := (a \leq i \wedge b > j) \vee (a < i \wedge b \geq j) \tag{12a}$$

$$\rho'_2(a,b,i,j) := a > 1 \wedge \rho'_1(a,b,i,j) \tag{12b}$$

$$\rho'(i,j) := \rho(i,j) \wedge \neg\exists a,b[\rho'_2(a,b,i,j) \wedge \rho(a,b)] \tag{12c}$$

For all $x \in \Lambda$, define $F_1(x)$ as in Equation 13.

$$F_1(x) := \neg\exists i,j[j > i > 1 \wedge \rho'(i,j)] \wedge A(\phi_x) \tag{13}$$

$F_1(x)$ is intended to coincide with ϕ_x on any $(P_k, H_k, L_k) \in D$ (i.e. where the hypothesis is *not* externally negated). The term $j > i > 1$ in Equation 13 allows for the possibility that the premise P_k may be externally negated in the original dataset D.

Lemma 5. *For all $x \in \Lambda$ and all $\sigma \in (\Sigma')^*$ such that $|\sigma| \leq w$ and there does not exist $\eta \in N^*$ such that η is a subsequence of $\sigma_{2:}$: $[\phi_x](\sigma) \leftrightarrow [F_1(x)](f(\sigma))$*

Proof. By Lemma 3, $[\phi_x](\sigma) \leftrightarrow [A(\phi_x)](f(\sigma))$. By assumption, $\neg\exists i,j[j > i > 1 \wedge \rho'(i,j)]$ holds for all such $f(\sigma)$. ☐

We then define $A'(\phi_x)$ by replacing each predicate $Q'_a(p)$ in $A(\phi_x)$ (Equation 8) with $\beta(Q'_a(p))$, as defined in Equation 14 (where i and j are free count variables in $A'(\phi_x)$).

$$\beta_1(Q'_a(p)) := p < i \wedge Q'_a(p) \tag{14a}$$

$$\beta_2(Q'_a(p)) := p \geq i \wedge Q'_a(p + (j-i) + 1) \tag{14b}$$

$$\beta(Q'_a(p)) := \beta_1(Q'_a(p)) \vee \beta_2(Q'_a(p)) \tag{14c}$$

Lemma 6. *For all $(P_k, H_k, L_k) \in D$, all $x \in \Lambda$, and all $\eta \in N^*$ such that $|P_k \eta H_k| \leq w$: $[\phi_x](P_k H_k) \leftrightarrow [A'(\phi_x)](f(P_k \eta H_k))$ when the free variables $i = |P_k| + 1$, $j = |P_k \eta|$ in Equation 14.*

Proof. We first prove that $[A(\phi_x)](f(P_k H_k)) \leftrightarrow [A'(\phi_x)](f(P_k \eta H_k))$. Note that $A'(\phi_x)$ is constructed from $A(\phi_x)$ by replacing each instance of $Q'_a(p)$ with $\beta(Q'_a(p))$. It therefore suffices to prove that for all $a \in \Sigma'$ and all $1 \leq p \leq w$: $[Q'_a(p)](f(P_k H_k)) \leftrightarrow [\beta(Q'_a(p))](f(P_k \eta H_k))$.

If $p \leq |P_k|$, then $[Q'_a(p)](f(P_k H_k)) \leftrightarrow [\beta(Q'_a(p))](f(P_k \eta H_k))$ by definition (Equation 14). Otherwise, $[Q'_a(p)](f(P_k H_k)) \leftrightarrow [\beta(Q'_a(p))](f(P_k \eta H_k))$ if and only if $(P_k H_k)_p = (P_k \eta H_k)_{p+(j-i)+1}$. By assumption, $p+(j-i)+1 = p+(|P_k\eta|-(|P_k|+1))+1 = p+|\eta|$ and $(P_k H_k)_p = (P_k \eta H_k)_{p+|\eta|}$.

By Lemma 3 and the above result, we have: $[\phi_x](P_i H_i) \leftrightarrow [A(\phi_x)](f(P_i H_i)) \leftrightarrow [A'(\phi_x)](f(P_i \eta H_i))$. □

Now, define $F_2(x)$ as in Equation 15, where $G(\mathcal{E}) = \mathcal{C}$, $G(\mathcal{C}) = \mathcal{E}$, and $G(\mathcal{N}) = \mathcal{N}$.

$$\gamma_x^1(n) := MODC_2(1, n) \wedge A'(\phi_{G(x)}) \tag{15a}$$

$$\gamma_x^2(n) := MODC_2(0, n) \wedge A'(\phi_x) \tag{15b}$$

$$\gamma_x^3(k) := i \leq k \leq j \wedge \psi(k) \tag{15c}$$

$$\gamma_x^4(n) := E(k, n, \gamma_x^3(k)) \tag{15d}$$

$$\gamma_x^5(n) := \neg\exists y[y > n \wedge E(k', y, \gamma_x^3(k'))] \tag{15e}$$

$$\gamma_x := \exists n[\gamma_x^4(n) \wedge \gamma_x^5(n) \wedge (\gamma_x^1(n) \vee \gamma_x^2(n))] \tag{15f}$$

$$F_2(x) := \exists i, j[j > i > 1 \wedge \rho'(i, j) \wedge \gamma_x] \tag{15g}$$

Lemma 7. *Define $N_0, N_1 \subset N^*$ as the sets of even- and odd-length (in terms of number of prefixes, rather than characters) sequences of external negation prefixes, respectively. Then for all $x \in \Lambda$ and all $(P_k, H_k, L_k) \in D$:*

i. for all $\eta \in N_0$: $[\phi_x](P_k H_k) \leftrightarrow [F_2(x)](f(P_k \eta H_k))$

ii. for all $\eta' \in N_1$: $[\phi_{G(x)}](P_k H_k) \leftrightarrow [F_2(x)](f(P_k \eta' H_k))$

Proof. We first prove (i). By Lemma 4 and the definition of $\rho'(i, j)$ (Equation 12), the respective values of i, j that make the term $j > i > 1 \wedge \rho'(i, j)$ hold in Equation 15 are $i = |P_k| + 1$ and $j = |P_k \eta|$. By the definitions of $E(k, n, -)$, $\psi(-)$, and γ_x (Equations 6, 10, and 15, respectively)—and the assumption that $\eta \in N_0$—the value of n that makes $[\gamma_x](f(P_k \eta H_k))$ hold is even. Therefore, the term $MODC_2(0, n)$ in $\gamma_x^2(n)$ holds, and so $[A'(\phi_x)](f(P_k \eta H_k)) \leftrightarrow [F_2(x)](f(P_k \eta H_k))$.

By Lemma 6 and the above result: $[\phi_x](P_k H_k) \leftrightarrow [A'(\phi_x)](f(P_k \eta H_k)) \leftrightarrow [F_2(x)](f(P_k \eta H_k))$.

We now prove (ii); the proof proceeds in a similar fashion as that of (i) above. But now n is odd, and so the term $MODC_2(1, n)$ in $\gamma_x^1(n)$ holds. Therefore, $[A'(\phi_{G(x)})](f(P_k \eta' H_k)) \leftrightarrow [F_2(x)](f(P_k \eta' H_k))$.

Again by Lemma 6 and the above result: $[\phi_{G(x)}](P_k H_k) \leftrightarrow [A'(\phi_{G(x)})](f(P_k \eta H_k)) \leftrightarrow [F_2(x)](f(P_k \eta H_k))$. □

For all $x \in \Lambda$, we define $F(x)$ as follows (Equation 16).

$$F(x) := F_1(x) \vee F_2(x) \tag{16}$$

We may now define the formula $S_{T'}$ in Equation 17 below.

$$S_{T'} := \bigwedge_{x \in \Lambda} F(x) \leftrightarrow \exists^{=1} p. Q_x(p) \tag{17}$$

Lemma 8. *For all $(P_k, H_k, L_k) \in D$, all $\eta \in N_0$ such that $|P_k \eta H_k| \leq w$, and all $\eta' \in N_1$ such that $|P_k \eta' H_k| \leq w$:*

i. $[S_{T'}](f(P_k H_k) L_k) \leftrightarrow [S_T](P_k H_k L_k)$

ii. $[S_{T'}](f(P_k \eta H_k) L_k) \leftrightarrow [S_T](P_k H_k L_k)$

iii. $[S_{T'}](f(P_k \eta' H_k) G(L_k)) \leftrightarrow [S_T](P_k H_k L_k)$

Proof. By Lemma 5, $[F_1(L_k)](f(P_k H_k))$ holds if and only if $[\phi_{L_k}](P_k H_k)$ does as well, for all $(P_k, H_k, L_k) \in D$. $F_2(x)$ does not hold for any $x \in \Lambda$ by definition (Equation 15), and $[F_1(x)](f(P_k H_k)) \leftrightarrow [\phi_x](P_k H_k)$ for any $x \in \Lambda - \{L_k\}$ by Lemma 5. This proves (i).

For all $\eta \in N_0$ such that $|P_k \eta H_k| \leq w$, $[F_1(x)](f(P_k H_k))$ does not hold for any $x \in \Lambda$ by definition (Equation 13), and $[F_2(x)](f(P_k H_k)) \leftrightarrow [\phi_x](P_k H_k)$ for all $x \in \Lambda$ by Lemma 7(i). This proves (ii).

For all $\eta' \in N_1$ such that $|P_k \eta' H_k| \leq w$, $[F_1(x)](f(P_k H_k))$ does not hold for any $x \in \Lambda$ by definition, and $[F_2(x)](f(P_k H_k)) \leftrightarrow [\phi_{G(x)}](P_k H_k)$ for all $x \in \Lambda$ by Lemma 7(ii). This proves (iii). □

By Chiang et al. (2023) Theorem 5, there exists a transformer encoder T'' that recognizes the language defined by $S_{T'}$. By Lemma 8(i), $Acc(T'', f(D)) = Acc(T, D)$, and $Acc(T'', \{f(P_i \eta H_i)\}_{i \in I}) = Acc(T, D)$ for any $\eta \in N^*$ such that $max_{i \in I} |P_i \eta H_i| \leq w$ by Lemma 8(ii-iii).

But T'' is an arbitrary-precision transformer. It remains to show that we can derive a *fixed*-precision transformer T' from T''. Note that by definition (Equation 2), for any $\sigma \in (\Sigma')^*$ such that $|\sigma| < w$: $|f(\sigma)| = w(|\sigma| + 1)$. By assumption (Theorem 1), no input example (challenge or otherwise) exceeds the fixed (finite) $w > max_{i \in I} |P_i H_i|$ in length. It follows that the upper bound on the length of possible inputs to T'' (within the assumptions of Theorem 1) is $w^2 + w$.

By definition, the floating-point precision of an arbitrary-precision transformer varies as a function of input length. Let $\pi \colon \mathbb{N} \to \mathbb{N}$ be the function mapping input length to floating-point precision (in bits) of T''—presumably, π is monotone-increasing, but it need not be. Define T' as T'' with floating-point precision fixed at $max_{1 \leq n \leq w^2+w} \pi(n)$.

This completes the proof of Theorem 1.

B Experiment 1

B.1 Deriving Class Labels

Given a premise, hypothesis, label triple (P, H, L) in an NLI dataset, the label L is defined as follows (where $\mathcal{E}$, $\mathcal{C}$, and $\mathcal{N}$ denote *entailment*, *contradiction*, and *neutral*, respectively):

$$L = \mathcal{E} \Leftrightarrow P \to H \tag{18a}$$

$$L = \mathcal{C} \Leftrightarrow P \to \neg H \tag{18b}$$

$$L = \mathcal{N} \Leftrightarrow L \neq \mathcal{E} \wedge L \neq \mathcal{C} \tag{18c}$$

Where the left-hand sides of the bidirectional arrows in Equation 18a-b hold in every logically possible state of affairs. Now consider the triple (P, H', L'), where $H' = \neg H$. If $L = \mathcal{E}$, then we have:

$$P \to H = P \to \neg\neg H = P \to \neg H' \Leftrightarrow L' = \mathcal{C} \tag{19}$$

By the law of the excluded middle, the definition of H', and Equation 18b. Therefore, $L = \mathcal{E} \leftrightarrow L' = \mathcal{C}$. Similarly, if $L = \mathcal{C}$, then we have:

$$P \to \neg H = P \to H' \Leftrightarrow L' = \mathcal{E} \tag{20}$$

By the definition of H' and Equation 18a. Therefore, $L = \mathcal{C} \leftrightarrow L' = \mathcal{E}$.

Now suppose that $L = \mathcal{N}$. By the above discussion (Equations 19 and 20), we have $L = \mathcal{E} \leftrightarrow L' = \mathcal{C}$ and $L = \mathcal{C} \leftrightarrow L = \mathcal{E}$. Therefore $L' \notin \{\mathcal{E}, \mathcal{C}\}$ and so (by Equation 18c) $L' = \mathcal{N}$. So we have $L = \mathcal{N} \to L' = \mathcal{N}$. Swapping L and L' in the above discussion in this paragraph, we have $L' = \mathcal{N} \to L = \mathcal{N}$. Therefore, $L = \mathcal{N} \leftrightarrow L' = \mathcal{N}$.

Template	**Premise**	**Hypothesis**	**Label**
(P, H)	A young boy dressed in plaid about to take a picture.	A young boy is about to take a picture.	Entailment
$(P, (T_{NT})^1 H)$		It is not true that a young boy is about to take a picture.	Contradiction
$(P, (T_{NT})^2 H)$		It is not true that it is not true that a young boy is about to take a picture.	Entailment
$(P, (T_{NT})^3 H)$		It is not true that it is not true that it is not true that a young boy is about to take a picture.	Contradiction
(P, H)	A race between friends at the park.	The park is deserted.	Contradiction
$(P, (T_{NT})^1 H)$		It is not true that the park is deserted.	Entailment
$(P, (T_{NT})^2 H)$		It is not true that it is not true that the park is deserted.	Contradiction
$(P, (T_{NT})^3 H)$		It is not true that it is not true that it is not true that the park is deserted.	Entailment
(P, H)	People kneeling on the ground.	People are praying.	Neutral
$(P, (T_{NT})^1 H)$		It is not true that people are praying.	Neutral

Table 5: Examples of depth-n negated challenge data points $(P, (T_{NT})^n H)$ generated from SNLI (some examples have been slightly modified for presentability).

B.2 Inoculation Development Set Accuracies

Model	Initial Acc. (Original)	Initial Acc. (Challenge)	Inoculated Acc. (Original)	Inoculated Acc. (Challenge)
$BART_M$	0.89	0.52	0.77	0.94
$RoBERTa_M$	0.89	0.51	0.87	0.93
$DeBERTa_S$	0.9	0.39	0.9	0.91
$RoBERTa_S$	0.88	0.57	0.88	0.89
$BART_{SMFA}$	0.89	0.69	0.87	0.92
$RoBERTa_{SMFA}$	0.87	0.51	0.86	0.91

Table 6: Model accuracy on the original and challenge development sets before and after depth-1 inoculation.

Model	Initial Acc. (Original)	Initial Acc. (Challenge)	Inoculated Acc. (Original)	Inoculated Acc. (Challenge)
$BART_M$	0.89	0.61	0.86	0.94
$RoBERTa_M$	0.89	0.63	0.87	0.97
$DeBERTa_S$	0.9	0.48	0.9	0.96
$RoBERTa_S$	0.88	0.66	0.88	0.94
$BART_{SMFA}$	0.89	0.72	0.88	0.95
$RoBERTa_{SMFA}$	0.87	0.65	0.88	0.95

Table 7: Model accuracy on the original and challenge development sets before and after depth-$\leq$2 inoculation.

Model	Initial Acc. (Original)	Initial Acc. (Challenge)	Inoculated Acc. (Original)	Inoculated Acc. (Challenge)
$BART_M$	0.89	0.53	0.87	0.95
$RoBERTa_M$	0.89	0.54	0.87	0.96
$DeBERTa_S$	0.9	0.45	0.9	0.96
$RoBERTa_S$	0.88	0.57	0.88	0.93
$BART_{SMFA}$	0.89	0.6	0.76	0.93
$RoBERTa_{SMFA}$	0.87	0.54	0.88	0.94

Table 8: Model accuracy on the original and challenge development sets before and after depth-$\leq$3 inoculation.

Model	Initial Acc. (Original)	Initial Acc. (Challenge)	Inoculated Acc. (Original)	Inoculated Acc. (Challenge)
$BART_M$	0.89	0.61	0.62	0.75
$RoBERTa_M$	0.89	0.62	0.74	0.88
$DeBERTa_S$	0.9	0.54	0.89	0.76
$RoBERTa_S$	0.88	0.64	0.89	0.89
$BART_{SMFA}$	0.89	0.66	0.62	0.86
$RoBERTa_{SMFA}$	0.87	0.61	0.88	0.89

Table 9: Model accuracy on the original and challenge development sets before and after depth-$\leq$4 inoculation.

Model	Initial Acc. (Original)	Initial Acc. (Challenge)	Inoculated Acc. (Original)	Inoculated Acc. (Challenge)
$BART_M$	0.89	0.55	0.32	0.74
$RoBERTa_M$	0.89	0.56	0.88	0.93
$DeBERTa_S$	0.9	0.5	0.9	0.91
$RoBERTa_S$	0.88	0.58	0.88	0.89
$BART_{SMFA}$	0.89	0.59	0.87	0.88
$RoBERTa_{SMFA}$	0.87	0.54	0.86	0.87

Table 10: Model accuracy on the original and challenge development sets before and after depth-$\leq$5 inoculation.

B.3 Post-Inoculation Test Accuracy

Depth-m test	No inoc.	Depth-1 inoc.	Depth-$\leq$2 inoc.	Depth-$\leq$3 inoc.
2	0.71	0.32	—	—
3	0.36	0.93	0.31	—
4	0.82	0.36	0.94	0.31
5	0.33	0.88	0.31	0.94
6	0.86	0.41	0.94	0.31

Table 11: Accuracy for $BART_M$ on depth-(m>n) external negation after depth-$\leq n$ inoculation ($n \in \{1, 2, 3\}$).

Depth-m test	No inoc.	Depth-1 inoc.	Depth-$\leq$2 inoc.	Depth-$\leq$3 inoc.
2	0.77	0.36	—	—
3	0.34	0.89	0.33	—
4	0.85	0.33	0.97	0.32
5	0.32	0.88	0.33	0.95
6	0.89	0.34	0.97	0.33

Table 12: Accuracy for $RoBERTa_M$ on depth-(m>n) external negation after depth-$\leq n$ inoculation ($n \in \{1, 2, 3\}$).

Depth-m test	No inoc.	Depth-1 inoc.	Depth-$\leq$2 inoc.	Depth-$\leq$3 inoc.
2	0.56	0.62	—	—
3	0.4	0.61	0.32	—
4	0.84	0.64	0.96	0.5
5	0.3	0.51	0.32	0.96
6	0.88	0.77	0.96	0.36

Table 13: Accuracy for $DeBERTa_S$ on depth-(m>n) external negation after depth-$\leq n$ inoculation ($n \in \{1, 2, 3\}$).

Depth-m test	No inoc.	Depth-1 inoc.	Depth-$\leq$2 inoc.	Depth-$\leq$3 inoc.
2	0.74	0.32	—	—
3	0.4	0.89	0.3	—
4	0.84	0.35	0.94	0.39
5	0.34	0.88	0.3	0.74
6	0.83	0.33	0.93	0.53

Table 14: Accuracy for $RoBERTa_S$ on depth-(m>n) external negation after depth-$\leq n$ inoculation ($n \in \{1, 2, 3\}$).

Depth-m test	No inoc.	Depth-1 inoc.	Depth-$\leq$2 inoc.	Depth-$\leq$3 inoc.
2	0.77	0.37	—	—
3	0.33	0.91	0.31	—
4	0.84	0.34	0.94	0.29
5	0.3	0.85	0.31	0.92
6	0.86	0.41	0.94	0.28

Table 15: Accuracy for $BART_{SMFA}$ on depth-(m>n) external negation after depth-$\leq n$ inoculation ($n \in \{1, 2, 3\}$).

Depth-m test	No inoc.	Depth-1 inoc.	Depth-$\leq$2 inoc.	Depth-$\leq$3 inoc.
2	0.79	0.35	—	—
3	0.32	0.93	0.32	—
4	0.83	0.31	0.95	0.32
5	0.32	0.94	0.32	0.94
6	0.84	0.32	0.95	0.32

Table 16: Accuracy for $RoBERTa_{SMFA}$ on depth-($m>n$) external negation after depth-$\leq n$ inoculation ($n \in \{1, 2, 3\}$).

C Experiment 3

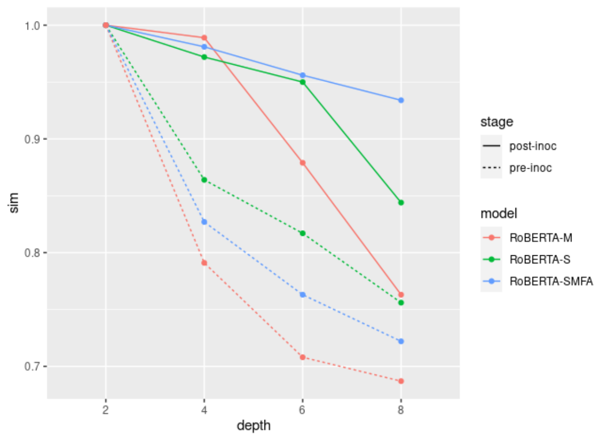

Figure 4: Mean cosine similarity between $(T_{NT})^n H_i$ and $(T_{NT})^2 H_i$ for the three RoBERTa models before (dashed) and after (solid) depth-$\leq$5 T_{NT} inoculation.

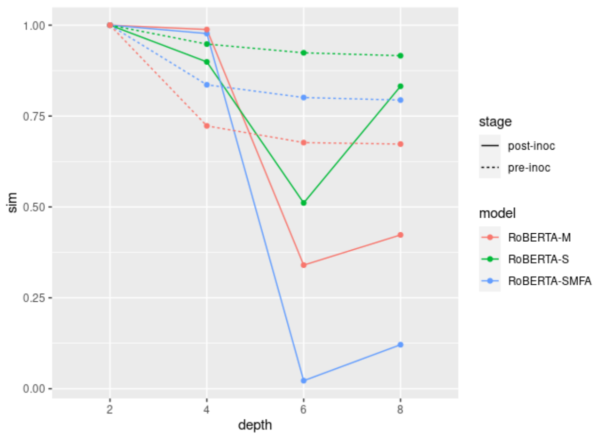

Figure 6: Mean cosine similarity between $(T_F)^n H_i$ and $(T_F)^2 H_i$ for the three RoBERTa models before (dashed) and after (solid) depth-$\leq$5 T_{NT} inoculation.

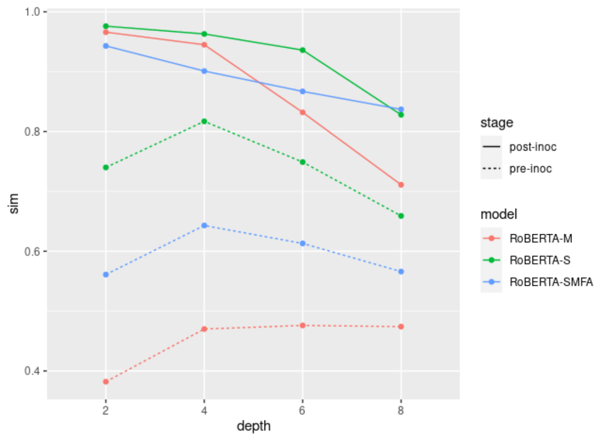

Figure 5: Mean cosine similarity between $(T_{NT})^n H_i$ and H_i for the three RoBERTa models before (dashed) and after (solid) depth-$\leq$5 T_{NT} inoculation.

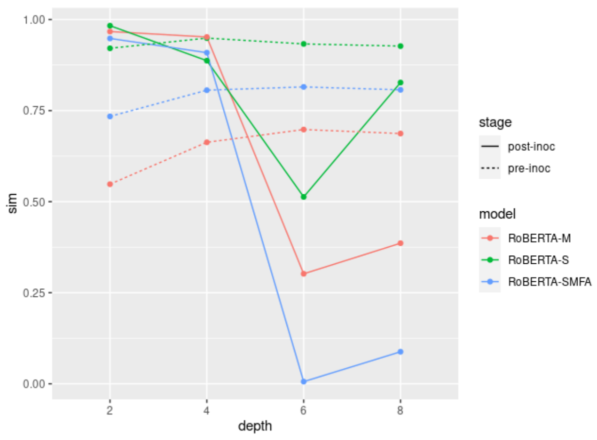

Figure 7: Mean cosine similarity between $(T_F)^n H_i$ and H_i for the three RoBERTa models before (dashed) and after (solid) depth-$\leq$5 T_{NT} inoculation.

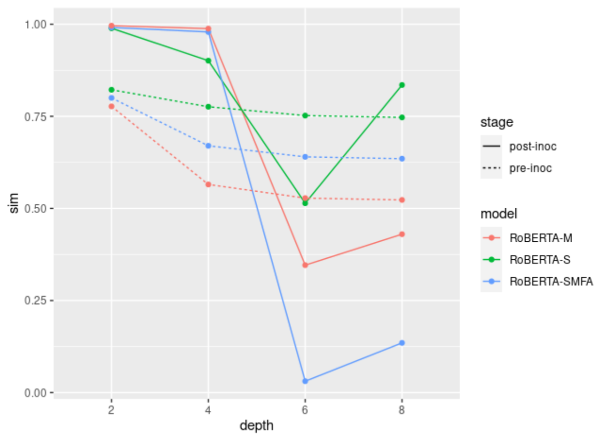

Figure 8: Mean cosine similarity between $(T_F)^n H_i$ and $(T_{NT})^2 H_i$ for the three RoBERTa models before (dashed) and after (solid) depth-≤ 5 T_{NT} inoculation.

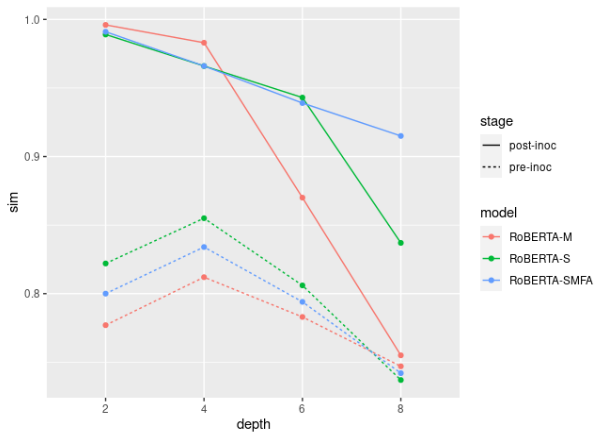

Figure 9: Mean cosine similarity between $(T_{NT})^n H_i$ and $(T_F)^2 H_i$ for the three RoBERTa models before (dashed) and after (solid) depth-≤ 5 T_{NT} inoculation.

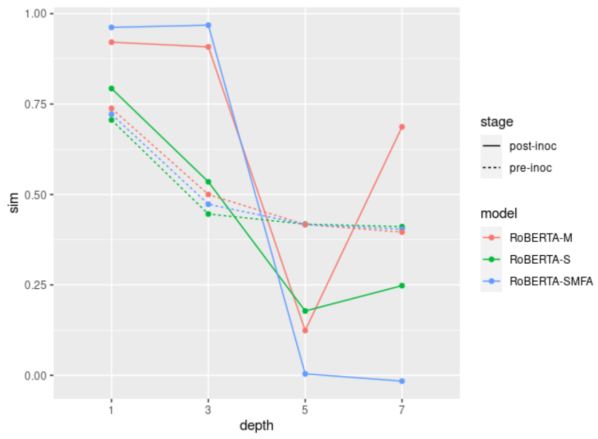

Figure 10: Mean cosine similarity between $(T_F)^n H_i$ and $(T_{NT})^1 H_i$ for the three RoBERTa models before (dashed) and after (solid) depth-≤ 5 T_{NT} inoculation.

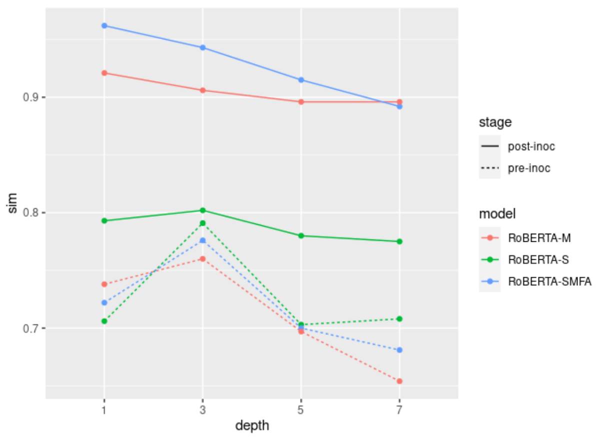

Figure 11: Mean cosine similarity between $(T_{NT})^n H_i$ and $(T_F)^1 H_i$ for the three RoBERTa models before (dashed) and after (solid) depth-≤ 5 T_{NT} inoculation.

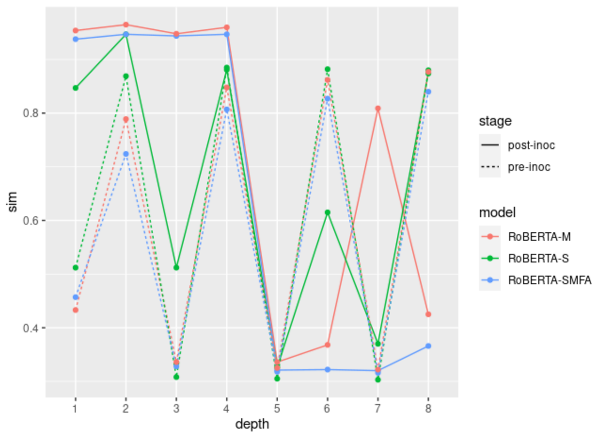

Figure 12: Mean cosine similarity between $(T_{NT})^n H_i$ and $(T_F)^n H_i$ for the three RoBERTa models before (dashed) and after (solid) depth-≤ 5 T_{NT} inoculation.