Complexity of the Acquisition of Phonotactics in Optimality Theory

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Abstract

The problem of the acquisition of Phonotactics in OT is shown to be not tractable in its *strong* formulation, whereby constraints and generating function vary arbitrarily as inputs of the problem.

Tesar and Smolensky (1998) consider the basic ranking problem in Optimality Theory (OT). According to this problem, the learner needs to find a ranking consistent with a given set of data. They show that this problem is solvable even in its *strong* formulation, namely without any assumptions on the generating function or the constraint set. Yet, this basic ranking problem is too simple to realistically model any actual aspect of language acquisition. To make the problem more realistic, we might want, for instance, to require the learner to find not just *any* ranking consistent with the data, rather one that furthermore generates a *smallest* language (w.r.t. set inclusion). Prince and Tesar (2004) and Hayes (2004) note that this computational problem models the task of the acquisition of phonotactics within OT. This paper shows that, contrary to the basic ranking problem considered by Tesar and Smolensky, this more realistic problem of the acquisition of phonotactics is *not* solvable, at least not in its strong formulation. I conjecture that this complexity result has nothing to do with the choice of the OT framework, namely that an analogous result holds for the corresponding problem within alternative frameworks, such as Harmonic Grammar (Legendre et al., 1990b; Legendre et al., 1990a). Furthermore, I conjecture that the culprit lies with the fact that generating function and constraint set are completely unconstrained. From this perspective, this paper motivates the following research question: to find phonologically plausible assumptions on generating function and constraint set that make the problem of the acquisition of phonotactics tractable.

1 Statement of the main result

Let the *universal specifications* of an OT typology be a 4-tuple $(\mathcal{X}, \mathcal{Y}, Gen, \mathcal{C})$, as illustrated in (1): $\mathcal{X}$ and $\mathcal{Y}$ are the sets of *underlying* and *surface* forms; Gen is the *generating function*; and $\mathcal{C}$ is the *constraint set*.

$$\begin{aligned} \mathcal{X} = \mathcal{Y} &= \{\text{ta, da, rat, rad}\} \\ Gen &= \big[\,\text{ta, da} \rightarrow \{\text{ta, da}\}\ \text{rat, rad} \rightarrow \{\text{rat, rad}\}\,\big] \\ \mathcal{C} &= \left\{ \begin{array}{c} F_{\text{pos}} = \text{IDNT[VCE]/ONSET}, \\ F = \text{IDNT[VCE]}, \\ M = {}^*[\text{+VCE}, -\text{SON}] \end{array} \right\} \end{aligned} \quad (1)$$

Let $\gg, \gg', \ldots$ be *rankings* over the constraint set, as illustrated in (2) for the constraint set in (1).

$$F_{\text{pos}} \gg M \gg F \qquad F_{\text{pos}} \gg' F \gg' M \quad (2)$$

Let $\mathsf{OT}_{\gg}$ be the *OT-grammar* corresponding to a ranking $\gg$ (Prince and Smolensky, 2004), as illustrated in (3) for the ranking $\gg$ in (2).

$$\begin{array}{cc} \mathsf{OT}_{\gg}(\text{/ta/}) = [\text{ta}] & \mathsf{OT}_{\gg}(\text{/da/}) = [\text{da}] \\ \mathsf{OT}_{\gg}(\text{/rat/}) = [\text{rat}] & \mathsf{OT}_{\gg}(\text{/rad/}) = [\text{rat}] \end{array} \quad (3)$$

Let $\mathcal{L}(\gg)$ be the *language* corresponding to a ranking $\gg$, illustrated in (4) for the rankings (2).

$$\begin{aligned} \mathcal{L}(\gg) &= \{\text{ta, da, rat}\} \\ \mathcal{L}(\gg') &= \{\text{ta, da, rat, rad}\} \end{aligned} \quad (4)$$

A *data set* $\mathcal{D}$ is a finite set of pairs $(x, \hat{y})$ of an underlying form $x \in \mathcal{X}$ and an intended winner surface form $\hat{y} \in Gen(x) \subseteq \mathcal{Y}$, as illustrated in (5).

$$\mathcal{D} = \{(\text{/da/}, [\text{da}]),\ (\text{/rat/}, [\text{rat}])\} \quad (5)$$

A data set $\mathcal{D}$ is called *OT-compatible with a ranking* $\gg$ iff the corresponding OT-grammar accounts for all the pairs in $\mathcal{D}$, namely $\mathsf{OT}_{\gg}(x) = \hat{y}$ for every pair $(x, \hat{y}) \in \mathcal{D}$. A data set $\mathcal{D}$ is called *OT-compatible* iff it is OT-compatible with at least a ranking. Suppose that the actual universal specifications $(\mathcal{X}, \mathcal{Y}, Gen, \mathcal{C})$ are fixed and known. The

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basic *Ranking problem* (Rpbm) is (6). The learner is provided with a set of data $\mathcal{D}$ corresponding to some target language; and has to come up with a ranking compatible with those data $\mathcal{D}$.

$$\begin{array}{rl} \textit{given:} & \text{an OT-comp. data set } \mathcal{D} \subseteq \mathcal{X} \times \mathcal{Y}; \\ \textit{find:} & \text{a ranking } \gg \text{ over the constraint set} \\ & \mathcal{C} \text{ that is OT-compatible with } \mathcal{D}. \end{array} \quad (6)$$

At the current stage of the development of the field, we have no firm knowledge of the actual universal specifications. Thus, the Rpbm (6) is of little interest. It is standard practice in the OT computational literature to get around this difficulty by switching to the *strong formulation* (7), whereby the universal specifications vary arbitrarily as an input to the problem (Wareham, 1998; Eisner, 2000; Heinz et al., 2009). Switching from (6) to (7) presupposes that the learner does not rely on peculiar properties of the actual universal specifications.

$$\begin{array}{rl} \textit{given:} & \text{univ. specs } (\mathcal{X}, \mathcal{Y}, Gen, \mathcal{C}), \\ & \text{an OT-comp. data set } \mathcal{D} \subseteq \mathcal{X} \times \mathcal{Y}; \\ \textit{find:} & \text{a ranking } \gg \text{ over the constraint set} \\ & \mathcal{C} \text{ that is OT-compatible with } \mathcal{D}. \end{array} \quad (7)$$

To complete the statement of the Rpbm (7), we need to specify the *size* of its instances, that determines the time that a solution algorithm is allowed to take. Let $width(\mathcal{D})$ be the cardinality of the largest candidate set over all underlying forms that appear in $\mathcal{D}$, as stated in (8).

$$width(\mathcal{D}) \stackrel{\text{def}}{=} \max_{(x,\hat{y}) \in \mathcal{D}} |Gen(x)| \quad (8)$$

Of course, the size of an instance of the Rpbm (7) depends on the cardinality $|\mathcal{C}|$ of the constraint set and on the cardinality $|\mathcal{D}|$ of the data set. Tesar and Smolensky (1998) (implicitly) assume that it also depends on $width(\mathcal{D})$, as stated in (9). [1]

$$\begin{array}{rl} \textit{given:} & \text{univ. specs } (\mathcal{X}, \mathcal{Y}, Gen, \mathcal{C}), \\ & \text{an OT-comp. data set } \mathcal{D} \subseteq \mathcal{X} \times \mathcal{Y}; \\ \textit{find:} & \text{a ranking } \gg \text{ of the constraint set } \mathcal{C} \\ & \text{that is OT-compatible with } \mathcal{D}; \\ \textit{size:} & \max\{|\mathcal{C}|,\ |\mathcal{D}|,\ width(\mathcal{D})\}. \end{array} \quad (9)$$

[1] A potential difficulty with the latter assumption is as follows: $width(\mathcal{D})$ could be very large, namely superpolynomial in the number of constraints $|\mathcal{C}|$; thus, letting the size of an instance of the Rpbm depend on $width(\mathcal{D})$ might make the problem too easy, by loosening up too much the tight dependence on $|\mathcal{C}|$. Yet, this potential difficulty is harmless in the case of the strong formulation of the Rpbm, since that formulation requires an algorithm to work for *any* universal specifications, and thus also for universal specifications where $|\mathcal{C}|$ is large but $width(\mathcal{D})$ small.

Tesar and Smolensky (1998) prove claim 1. This claim is important because it shows that no harm comes from switching to the strong formulation, at least in the case of the Rpbm.

Claim 1 *The Rpbm (9) is tractable.*

Yet, the Rpbm (9) is way too simple to realistically model any aspect of acquisition. Here is a way to appreciate this point. The two rankings $\gg$ and $\gg'$ in (2) are both solutions of the instance of the Rpbm (9) corresponding to the universal specifications in (1) and to the data set in (5). As noted in (4), the language corresponding to $\gg$ is a proper subset of the language corresponding to $\gg'$. A number of authors have suggested that the ranking $\gg$ that corresponds to the subset language is a "better" solution than the ranking $\gg'$ that corresponds to the superset language (Berwick, 1985; Manzini and Wexler, 1987; Prince and Tesar, 2004; Hayes, 2004). This intuition is captured by problem (10): it asks not just for *any* ranking OT-compatible with the data $\mathcal{D}$; rather, for one such ranking whose corresponding language is as small as possible (w.r.t. set inclusion). The latter condition requires the learner to rule out as *illicit* any form which is not entailed by the data. Problem (10) thus realistically models the task of the acquisition of phonotactics, namely the knowledge of licit vs. illicit forms.

$$\begin{array}{rl} \textit{given:} & \text{univ. specs } (\mathcal{X}, \mathcal{Y}, Gen, \mathcal{C}), \\ & \text{an OT-comp. data set } \mathcal{D} \subseteq \mathcal{X} \times \mathcal{Y}; \\ \textit{find:} & \text{a ranking } \gg \text{ OT-comp. with } \mathcal{D} \text{ s.t.} \\ & \text{there is no ranking } \gg' \text{ OT-comp.} \\ & \text{with } \mathcal{D} \text{ too s.t. } \mathcal{L}(\gg') \subsetneqq \mathcal{L}(\gg). \end{array} \quad (10)$$

The *Problem of the Acquisition of Phonotactics* (APpbm) in (10) involves the language $\mathcal{L}(\gg)$, which in turn depends on the number of forms in $\mathcal{X}$ and on the cardinality of the candidate set $Gen(x)$ for all underlying forms $x \in \mathcal{X}$. Thus, (11) lets the size of an instance of the APpbm depend generously on $|\mathcal{X}|$ and $width(\mathcal{X})$, rather than on $|\mathcal{D}|$ and $width(\mathcal{D})$ as in the case of the Rpbm (9).[2]

$$\begin{array}{rl} \textit{given:} & \text{univ. specs } (\mathcal{X}, \mathcal{Y}, Gen, \mathcal{C}), \\ & \text{an OT-comp. data set } \mathcal{D} \subseteq \mathcal{X} \times \mathcal{Y}; \\ \textit{find:} & \text{a ranking } \gg \text{ OT-comp. with } \mathcal{D} \text{ s.t.} \\ & \text{there is no ranking } \gg' \text{ OT-comp.} \\ & \text{with } \mathcal{D} \text{ too s.t. } \mathcal{L}(\gg') \subsetneqq \mathcal{L}(\gg); \\ \textit{size:} & \max\{|\mathcal{C}|,\ |\mathcal{X}|,\ width(\mathcal{X})\}. \end{array} \quad (11)$$

[2] Letting the size of an instance of the APpbm depend on $|\mathcal{C}|$, $|\mathcal{X}|$ and $width(\mathcal{X})$ ensures that the problem is in $\mathcal{NP}$, namely that it admits an efficient verification algorithm.

Prince and Tesar (2004) offer an alternative formulation of the APpbm. They define a *strictness measure* as a function μ that maps a ranking $\gg$ to a number $\mu(\gg)$ that provides a relative measure of the cardinality of the corresponding language $\mathcal{L}(\gg)$, in the sense that any solution of the problem (12) is a solution of the APpbm (10).[3]

$$\begin{array}{rl} \textit{given:} & \text{univ. specs } (\mathcal{X}, \mathcal{Y}, Gen, \mathcal{C}), \\ & \text{an OT-comp. data set } \mathcal{D} \subseteq \mathcal{X} \times \mathcal{Y}; \\ \textit{find:} & \text{a ranking with minimal measure } \mu \\ & \text{among those OT-comp. with } \mathcal{D}. \end{array} \tag{12}$$

As usual, assume that the constraint set $\mathcal{C}on = \mathcal{F} \cup \mathcal{M}$ is split up into the subset $\mathcal{F}$ of faithfulness constraints and the subset $\mathcal{M}$ of markedness constraints. Consider the function μ_{PT} defined in (13): it pairs a ranking $\gg$ with the number $\mu_{\mathrm{PT}}(\gg)$ of pairs of a faithfulness constraint and a markedness constraint such that the former is $\gg$-ranked above the latter. Prince and Tesar (2004) conjecture that the function μ_{PT} in (13) is a strictness measure. The intuition is that faithfulness (markedness) constraints work toward (against) preserving underlying contrasts and thus a small language is likely to arise by having few pairs of a faithfulness constraint ranked above a markedness constraint.

$$\mu_{\mathrm{PT}}(\gg) \overset{\mathrm{def}}{=} |\{(F, M) \in \mathcal{F} \times \mathcal{M} \mid F \gg M\}| \tag{13}$$

Let me dub (12) with the mesure μ_{PT} in (13) *Prince and Tesar's reformulation* of the APpbm (PTAPpbm), as in (14). The core idea of strictness measures is to determine the relative strictness of two rankings without reference to the entire set of forms $\mathcal{X}$. Thus, (14) lets the size of an instance of PTAPpbm depend on $|\mathcal{D}|$ and $width(\mathcal{D})$, rather than on $|\mathcal{X}|$ and $width(\mathcal{X})$ as for the APpbm (11).

$$\begin{array}{rl} \textit{given:} & \text{univ. specs } (\mathcal{X}, \mathcal{Y}, Gen, \mathcal{C}), \\ & \text{an OT-comp. data set } \mathcal{D} \subseteq \mathcal{X} \times \mathcal{Y}; \\ \textit{find:} & \text{a ranking with minimal measure} \\ & \mu_{\mathrm{PT}} \text{ among those OT-comp. with } \mathcal{D}; \\ \textit{size:} & \max\{|\mathcal{C}|, |\mathcal{D}|, width(\mathcal{D})\}. \end{array} \tag{14}$$

The APpbm (11) and the PTAPpbm (14) have figured prominently in the recent computational OT literature. The main result of this paper is claim 2. This claim says that there is no efficient algorithm for the APpbm nor for the PTAPpbm. I conjecture that the culprit lies in the switch to the strong formulation. Comparing with claim 1, I thus conclude that the switch is harmless for the easy Rpbm, but harmful for the more realistic APpbm and PTAPpbm.

Claim 2 *The APpbm (11) and the PTAPpbm (14) are intractable.*

In the next section, I prove NP-completeness of PTAPpbm by showing that the *Cyclic Ordering* problem can be reduced to PTAPpbm. I then prove NP-completeness of APpbm by showing that PTAPpbm can be reduced to it. NP-completeness of APpbm holds despite the generous dependence of its size on $|\mathcal{X}|$ and $width(\mathcal{X})$. Furthermore, the proof actually shows that the PTAPpbm remains NP-complete even when the data have the simplest "disjunctive structure", namely for each underlying/winner/loser form there are at most two winner-preferring constrains.[4] And furthermore even when the data have the property that the faithfulness constraints are never loser-preferring.

2 Proof of the main result

Given a data set $\mathcal{D}$, for every pair $(x, \hat{y}) \in \mathcal{D}$ of an underlying form x and a corresponding winner form $\hat{y}$, for every loser candidate $y \in Gen(x)$ different from $\hat{y}$, construct a row $\mathbf{a}$ with $|\mathcal{C}|$ entries as follows: the kth entry is an L if constraint C_k assigns more violations to the winner pair $(x, \hat{y})$ than to the loser pair (x, y); it is a W if the opposite holds; it is an E if the two numbers of violations coincide. Organize these rows one underneath the other into a tableau $\mathbf{A}(\mathcal{D})$, called the *comparative tableau corresponding to* $\mathcal{D}$. To illustrate, I give in (15) the tableau corresponding to the data set (5).

$$\mathbf{A}(\mathcal{D}) = \begin{array}{c} \begin{matrix} F & F_{\mathrm{pos}} & M \end{matrix} \\ \begin{bmatrix} \mathrm{W} & \mathrm{W} & \mathrm{L} \\ \mathrm{W} & \mathrm{E} & \mathrm{W} \end{bmatrix} \end{array} \tag{15}$$

Generalizing a bit, let $\mathbf{A} \in \{\mathrm{L}, \mathrm{E}, \mathrm{W}\}^{m \times n}$ be a tableau with m rows, n columns, and entries taken from the three symbols L, E or W, called a *comparative tableau*. Let me say that $\mathbf{A}$ is *OT-compatible* with a ranking $\gg$ iff the tableau obtained by reordering the columns of $\mathbf{A}$ from left-to-right in

[3] The Rpbm (7) corresponds to *Empirical Risk Minimization* in the Statistical Learning literature, while problem (12) corresponds to a *regularized* version thereof, with regularization function μ.

[4] Of course, if there were a unique winner-preferring constraint per underlying/winner/loser form triplet, then the data would be OT-compatible with a unique ranking, and thus the PTAPpbm would reduce to the Rpbm.

decreasing order according to $\gg$ has the property that the left-most entry different from E is a W in every row. Tesar and Smolensky (1998) note that a data set $\mathcal{D}$ is OT-compatible with a ranking $\gg$ iff the corresponding comparative tableau $\mathbf{A}(\mathcal{D})$ is OT-compatible with it. Thus, the PTAPpbm (14) is tractable iff the problem (16) is tractable. Note that this equivalence crucially depends on two facts. First, that the size of an instance of the PTAPpbm depends not only on $|\mathcal{C}|$ and $|\mathcal{D}|$, but also on $width(\mathcal{D})$. Second, that we are considering the strong formulation of the PTAPpbm, and thus no assumptions need to be imposed on the given comparative tableau in (16), besides it being OT-compatible. The set $\mathcal{F}$ provided with an instance of (16) says which one of the n columns of the comparative tableau $\mathbf{A}$ correspond to faithfulness constraints. The size of an instance of problem (16) of course depends on the numbers m and n of rows and columns of $\mathbf{A}$.

given: a OT-comp. tabl. $\mathbf{A} \in \{\text{L}, \text{E}, \text{W}\}^{m \times n}$, (16)
a set $\mathcal{F} \subseteq \{1, \ldots, n\}$;
find: a ranking $\gg$ with minimal measure μ_{PT} among those OT-comp. with $\mathbf{A}$;
size: $\max\{m,\, n\}$.

The decision problem corresponding to (16) is stated in (17). As it is well known, intractability of the decision problem (17) entails intractability of the original problem (16). In fact, if the original problem (16) can be solved in polynomial time, then the corresponding decision problem (17) can be solved in polynomial time too: given an instance of the decision problem (17), find a solution $\gg$ of the corresponding instance of (16) and then just check whether $\mu_{\text{PT}}(\gg) \leq k$. From now on, I will refer to (17) as the PTAPpbm.

given: a OT-comp. tabl. $\mathbf{A} \in \{\text{L}, \text{E}, \text{W}\}^{m \times n}$, (17)
a set $\mathcal{F} \subseteq \{1, \ldots, n\}$,
an integer k;
output: "yes" iff there is a ranking $\gg$ OT-comp. with $\mathbf{A}$ s.t. $\mu_{\text{PT}}(\gg) \leq k$;
size: $\max\{m,\, n\}$.

Let me now introduce the problem I will reduce to PTAPpbm. Given a finite set $A = \{a,\, b,\, \ldots\}$ with cardinality $|A|$, consider a set $S \subseteq A \times A$ of pairs of elements of A. The set S is called *linearly compatible* iff there exists a one-to-one function $\pi : A \rightarrow \{1, 2, \ldots, |A|\}$ such that for every pair $(a, b) \in S$ we have $\pi(a) < \pi(b)$. It is useful to let S be not just a set but a *multiset*, namely to allow S to contain multiple instances of the same pair. The notion of cardinality and the subset relation are trivially extended from sets to multisets. Consider the problem (18), that I will call the *Max-ordering problem* (MOpbm).

given: a finite set A, (18)
a multiset $P \subseteq A \times A$,
an integer $k \leq |P|$;
output: "yes" iff there is a linearly compatible multiset $S \subseteq P$ with $|S| \geq k$;
size: $\max\{|A|,\, |P|\}$.

The PTAPpbm (17) is clearly in $\mathcal{NP}$, namely it admits a verification algorithm. Claim 3 ensures that MOpbm (18) is NP-complete. Claim 4 shows that MOpbm can be reduced to PTAPpbm (17). I can thus conclude that PTAPpbm is NP-complete.

Claim 3 *The MOpbm (18) is NP-complete.*[5]

Proof. The MOpbm is obviously in $\mathcal{NP}$. To show that it is NP-complete, I need to exhibit an NP-complete problem that can be reduced to it. Given a finite set $A = \{a,\, b,\, \ldots\}$ with cardinality $|A|$, consider a set $T \subseteq A \times A \times A$ of triplets of elements of A. The set T is called *linearly cyclically compatible* iff there exists a one-to-one function $\pi : A \rightarrow \{1, 2, \ldots, |A|\}$ such that for every triplet $(a, b, c) \in T$ either $\pi(a) < \pi(b) < \pi(c)$ or $\pi(b) < \pi(c) < \pi(a)$ or $\pi(c) < \pi(a) < \pi(b)$. Consider the *Cyclic Ordering* problem (COpbm) in (19).[6] Galil and Megiddo (1977) prove NP-completeness of COpbm by reduction from the *3-Satisfability* problem; the COpbm is problem [MS2] in (Garey and Johnson, 1979, p. 279).

input: a finite set A; (19)
a set $T \subseteq A \times A \times A$;
output: "yes" iff T is linearly cyclically compatible;
size: $|A|$

Given an instance (A, T) of the COpbm (19), consider the corresponding instance (A, P, k) of the MOpbm (18) defined as in (20). For every triplet

[5] A similar claim appears in (Cohen et al., 1999).

[6] It makes sense to let the size of an instance of the COpbm (19) be just the cardinality of the set A. In fact, the cardinality of the set T can be at most $|A|^3$. On the other hand, it makes sense to let the size of an instance of the MOpbm (18) depend also on the cardinality of the multiset P rather than only on the cardinality of the set A, since P is a multiset and thus its cardinality cannot be bound by the cardinality of A.

(a, b, c) in the set T, we put in the multiset P the three pairs (a, b), (b, c) and (c, a). Furthermore, we set the threshold k to twice the number of triplets in the set T. Note that P is a multiset because it might contain two instances of the same pair coming from two different triplets in T.

$$\begin{aligned} P &= \Big\{(a,b),\ (b,c),\ (c,a) \,\Big|\, (a,b,c) \in T\Big\} \\ k &= 2|T| \end{aligned} \tag{20}$$

Assume that the instance (A, T) of the COpbm admits a positive answer. Thus, T is cyclically compatible with a linear order π on A. Thus, for every triplet $(a, b, c) \in T$, there are at least two pairs in P compatible with π. Hence, there is a multiset S of pairs of P with cardinality at least $k = 2|T|$ linearly compatible with π,[7] namely the instance of the MOpbm defined in (20) admits a positive answer. Vice versa, assume that the instance (A, P, k) of the MOpbm in (20) admits a positive answer. Thus, there exists a linear order π on A compatible with $2|T|$ pairs in P. Since the three pairs that come from a given triplet are inconsistent, then each triplet must contribute two pairs to the total of $2|T|$ compatible pairs. Hence, π is cyclically compatible with all triplets in T. ■

Claim 4 *The MOpbm (18) can be reduced to the PTAPpbm (17).*

Proof. Given an instance (A, P, k) of the MOpbm, construct the corresponding instance $(\mathbf{A}, \mathcal{F}, K)$ of the PTAPpbm as follows. Let $n = |A|$, $\ell = |P|$; pick an integer d as in (21).

$$d > (\ell - k)n \tag{21}$$

Let the threshold K and the numbers N and M of columns and rows of the tableau $\mathbf{A}$ be as in (22).

$$\begin{aligned} K &= (\ell - k)(n + d) \\ N &= \ell + n + d \\ M &= \ell + nd \end{aligned} \tag{22}$$

Let the sets $\mathcal{F}$ and $\mathcal{M}$ of faithfulness and markedness constraints be as in (23). There is a faithfulness constraint $F_{(i,j)}$ for every pair (a_i, a_j) in the multiset P in the given instance of the MOpbm. Markedness constraints come in two varieties. There are the markedness constraints $M_1, \ldots, M_n$, one for every element in the set A in the given instance of the MOpbm; and then there are d more markedness constraints $M'_1, \ldots, M'_d$, that I'll call the *ballast* markedness constraints.

$$\begin{aligned} \mathcal{F} &= \{F_{(i,j)} \,|\, (a_i, a_j) \in P\} \\ \mathcal{M} &= \{M_1, \ldots, M_n\} \cup \{M'_1, \ldots, M'_d\} \end{aligned} \tag{23}$$

The comparative tableau $\mathbf{A}$ is built by assembling one underneath the other various blocks. To start, let $\overline{\mathbf{A}}$ be the block with ℓ rows and $N = \ell+n+d$ columns described in (24). It has a row for every pair $(a_i, a_j) \in P$. This row has all E's but for three entries: the entry corresponding to the faithfulness constraint $F_{(i,j)}$ corresponding to that pair, which is a W; the entry corresponding to the markedness constraint M_i corresponding to the first element a_i in the pair, which is an L; the entry corresponding to the markedness constraint M_j corresponding to the second element a_j in the pair, which is a W.

$$\begin{array}{c} \quad\quad\quad \cdots\ F_{(i,j)}\ \cdots\quad \cdots\ M_i\ \cdots\ M_j\ \cdots\ M'_1\ \cdots\ M'_d \\ (a_i, a_j) \Rightarrow \left[\begin{array}{ccc|ccccc|ccc} & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots \\ \ldots & \mathrm{W} & \ldots & \ldots & \mathrm{L} & \ldots & \mathrm{W} & \ldots & \mathrm{E} & \ldots & \mathrm{E} \\ & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots \end{array}\right] \end{array} \tag{24}$$

Next, let $\mathbf{A}_i$ be the block with d rows and $N = \ell + n + d$ columns described in (25), for every $i = 1, \ldots, n$. All entries corresponding to the faithfulness constraints are equal to E. All entries corresponding to the the markedness constraints $M_1, \ldots, M_n$ are equal to E, but for those in the column corresponding to M_i, that are instead equal to W. All entries corresponding to the ballast constraints $M'_1, \ldots, M'_d$ are equal to E, but for the diagonal entries that are instead equal to L.

$$\begin{array}{c} F_1\ \cdots\ F_\ell\quad M_1\ \cdots\ M_i\ \cdots\ M_n\quad M'_1\ \cdots\ M'_d \\ \left[\begin{array}{ccc|c|ccc} \mathrm{E} & \ldots & \mathrm{E} & \mathrm{W} & \mathrm{L} & & \\ \vdots & & \vdots & | & & \diagdown & \\ \mathrm{E} & \ldots & \mathrm{E} & \mathrm{W} & & & \mathrm{L} \end{array}\right] \end{array} \tag{25}$$

Finally, let the comparative tableau $\mathbf{A}$ be obtained by ordering the $n + 1$ blocks $\overline{\mathbf{A}}, \mathbf{A}_1, \ldots, \mathbf{A}_n$ one underneath the other, as in (26). Before I turn to the details, let me present the intuition behind the

[7] Note that, in order for the latter claim to hold, it is crucial that P be a multiset, namely that the same pair might be counted twice. In fact, T might contain two different triplets that share some elements, such as (a, b, c) and (a, b, d).

definitions (21)-(26).

$$
\begin{array}{c|c|c|c|}
 & F_1 \;\dots\; F_\ell & M_1 \;\dots\; M_n & M'_1 \;\dots\; M'_d \\
\hline
 & \multicolumn{3}{c|}{\overline{\mathbf{A}}} \\
\hline
\mathbf{A}_1 & \begin{matrix} \mathrm{E} \dots \mathrm{E} \\ \vdots \quad\;\; \vdots \\ \mathrm{E} \dots \mathrm{E} \end{matrix} & \begin{matrix} \mathrm{W} & & \\ | & & \\ \mathrm{W} & & \end{matrix} & \begin{matrix} \mathrm{L} & & \\ & \ddots & \\ & & \mathrm{L} \end{matrix} \\
\hline
 & \vdots & \vdots & \vdots \\
\hline
\mathbf{A}_n & \begin{matrix} \mathrm{E} \dots \mathrm{E} \\ \vdots \quad\;\; \vdots \\ \mathrm{E} \dots \mathrm{E} \end{matrix} & \begin{matrix} & & \mathrm{W} \\ & & | \\ & & \mathrm{W} \end{matrix} & \begin{matrix} \mathrm{L} & & \\ & \ddots & \\ & & \mathrm{L} \end{matrix}
\end{array}
\tag{26}
$$

Since the markedness constraints $M_1, \dots, M_n$ correspond to the elements $a_1, \dots, a_n$ of A, a linear order π over A defines a ranking $\gg$ of the markedness constraint $M_1, \dots, M_n$ as in (27), and viceversa. Thus, π is linearly compatible with a pair $(a_i, a_j) \in P$ iff the row of the block $\overline{\mathbf{A}}$ in (24) corresponding to that pair is accounted for by ranking M_j above M_i, with no need for the corresponding faithfulness constraint $F_{(i,j)}$ to do any work. Suppose instead that M_j is not ranked above M_i, so that the corresponding faithfulness constraint $F_{(i,j)}$ needs to be ranked above M_i in order to protect its L. What consequences does this fact have for the measure μ_{PT} in (13)? Without the ballast constraints $M'_1, \dots, M'_d$, not much: all I could conclude is that the faithfulness constraint $F_{(i,j)}$ has at least the two markedness constraints M_i and M_j ranked below it. The ballast markedness constraints $M'_1, \dots, M'_d$ ensure a more dramatic effect. In fact, the block $\mathbf{A}_i$ forces each of them to be ranked below M_i. Thus, if the faithfulness constraint $F_{(i,j)}$ needs to be ranked above M_i, then it also needs to be ranked above all the ballast markedness constraints $M'_1, \dots, M'_d$. If the number d of these ballast constraints is large enough, as in (21), then the corresponding effect on the measure μ_{PT} in (13) is rather dramatic.

$$M_j \gg M_i \iff \pi(a_j) > \pi(a_i) \tag{27}$$

Assume that the given instance (A, P, k) of MOpbm admits a positive answer. Thus, there exists a multitset S of k pairs of P that is compatible with a linear order π on A. Consider a ranking $\gg$ over the constraint set (23) that satisfies the conditions in (28): $\gg$ assigns the k faithfulness constraints $F_{(i,j)}$ that correspond to pairs in S to the k bottom strata, in any order; $\gg$ assigns the d ballast markedness constraints $M'_1, \dots, M'_d$ to the next d strata, in any order; $\gg$ assigns the n markedness constraints $M_1, \dots, M_n$ to the next n strata, ordered according to π through (27); finally, $\gg$ assigns the remaining $\ell - k$ faithfulness constraints $F_{(i,j)}$ that correspond to pairs in $P \setminus S$ to the top $\ell - k$ strata, in any order.

$$
\begin{array}{c}
\{F_{(i,j)} \,|\, (a_i, a_j) \notin S\} \\
| \\
M_{\pi^{-1}(n)} \\
| \\
\vdots \\
| \\
M_{\pi^{-1}(1)} \\
| \\
\{M'_1, \dots, M'_d\} \\
| \\
\{F_{(i,j)} \,|\, (a_i, a_j) \in S\}
\end{array}
\tag{28}
$$

This ranking $\gg$ is OT-compatible with the comparative tableau $\mathbf{A}$ in (26). In fact, it is OT-compatible with the n blocks $\mathbf{A}_1, \dots, \mathbf{A}_n$ in (25), since the markedness constraints $M_1, \dots, M_n$ are $\gg$-ranked above the ballast markedness constraints $M'_1, \dots, M'_d$. It is OT-compatible with each row of the block $\overline{\mathbf{A}}$ in (24) that corresponds to a pair $(a_i, a_j) \notin S$, since the corresponding faithfulness constraint $F_{(i,j)}$ is $\gg$-ranked above the corresponding markedness constraints M_i. Finally, it is OT-compatible with each row of the block $\overline{\mathbf{A}}$ that corresponds to a pair $(a_i, a_j) \in S$, since $\pi(a_j) > \pi(a_i)$ and thus $M_j \gg M_i$ by (27). The measure $\mu_{\text{PT}}(\gg)$ of the ranking $\gg$ is (29): in fact, the faithfulness constraints $F_{(i,j)}$ corresponding to pairs $(a_i, a_j) \in S$ have no markedness constraints $\gg$-ranked below them; and each one of the $\ell - k$ faithfulness constraints $F_{(i,j)}$ corresponding to pairs $(a_i, a_j) \notin S$ has all the $n + d$ markedness constraints $\gg$-ranked below it. In conclusion, the instance $(\mathbf{A}, \mathcal{F}, K)$ of the PTAPpbm constructed in (21)-(26) admits a positive answer.

$$\mu_{\text{PT}}(\gg) = (\ell - k)(n + d) = K \tag{29}$$

Vice versa, assume that the instance $(\mathbf{A}, \mathcal{F}, K)$ of the PTAPpbm constructed in (21)-(26) admits a positive answer. Thus, there exists a ranking $\gg$ over the constraint set (23) OT-compatible with the tableau $\mathbf{A}$ in (26) such that $\mu_{\text{PT}}(\gg) \leq K$. Consider the multiset $S \subseteq P$ defined in (30). Clearly, S is compatible with the linear order π univocally defined on $A = \{a_1, \dots, a_n\}$ through (27).

$$S = \Big\{ (a_i, a_j) \in P \;\Big|\; M_j \gg M_i \Big\} \tag{30}$$

To prove that the given instance (A, P, k) of the MOpbm has a positive answer, I thus only need to show that $|S| \geq k$. Assume by contradiction that $|S| < k$. I can then compute as in (31). In step (31a), I have used the definition (22) of the threshold K. In step (31b), I have used the hypothesis that the ranking $\gg$ is a solution of the instance $(\mathbf{A}, \mathcal{F}, K)$ of the PTAPpbm and thus its mesure μ_{PT} does not exceed K. By (13), $\mu_{\text{PT}}(\gg)$ is the total number of pairs of a faithfulness constraint and a markedness constraint such that the former is $\gg$-ranked above the latter. In step (31c), I have thus lower bounded $\mu_{\text{PT}}(\gg)$ by only considering those faithfulness constraints $F_{(i,j)}$ corresponding to pairs (a_i, a_j) not in S. For each such constraint $F_{(i,j)}$, we have $M_i \gg M_j$, by the definition (30) of S. Thus, $F_{(i,j)}$ needs to be $\gg$-ranked above M_i in order to ensure OT-compatibility with the corresponding row of the block $\overline{\mathbf{A}}$ in (24). Since M_i needs to be $\gg$-ranked above the d ballast constraints $M'_1, \ldots, M'_d$ in order to ensure OT-compatibility with the block $\mathbf{A}_i$ in (25), then $F_{(i,j)}$ needs to be $\gg$-ranked above those d ballast markedness constraints too. In conclusion, each faithfulness constraint $F_{(i,j)}$ corresponding to a pair (a_i, a_j) not in S needs to be $\gg$-ranked at least above d markedness constraints. Since there are $\ell - |S|$ such faithfulness constraint $F_{(i,j)}$ corresponding to a pair $(a_i, a_j) \notin S$, then we get the inequality in (31d). In step (31e), I have used the absurd hypothesis that $|S| < k$ or equivalently that $|S| \leq k - 1$. The chain of inequalities in (31) entails that $d \leq (\ell - k)n$, which contradicts the choice (21) of the number d of ballast constraints.

$$
\begin{aligned}
&(\ell - k)d + (\ell - k)n \\
&\overset{(a)}{=} \quad K \\
&\overset{(b)}{\geq} \quad \mu_{\text{PT}}(\gg) \overset{(13)}{=} |\{(F_{(i,j)}, M) \mid F_{(i,j)} \gg M\}| \\
&\overset{(c)}{\geq} \quad |\{(F_{(i,j)}, M) \mid F_{(i,j)} \gg M,\ (a_i, a_j) \notin S\}| \\
&\overset{(d)}{=} \quad (\ell - |S|)d \\
&\overset{(e)}{\geq} \quad (\ell - (k-1))d \\
&= \quad (\ell - k)d + d
\end{aligned} \tag{31}
$$

The preceding considerations show that given an arbitrary instance (A, P, k) of the MOpbm (18), the corresponding instance $(\mathbf{A}, \mathcal{F}, K)$ of the PTAPpbm (17) defined in (21)-(26) admits a positive solution iff the original instance (A, P, k) of the MOpbm does. I conclude that the MOpbm can be reduced to the PTAPpbm. ■

Let me now turn to the APpbm (11). Once again, in order to show that it is intractable, it is sufficient to show that the corresponding decision problem (32) is intractable. In fact, if problem (11) can be solved, then (32) can be solved too: given an instance of the latter, find a solution $\gg$ of the corresponding instance of the problem (11) and then just check whether $|\mathcal{L}(\gg)| \leq k$.[8] From now on, I will refer to (32) as the APpbm.

given: univ. specs $(\mathcal{X}, \mathcal{Y}, Gen, \mathcal{C})$, an OT-comp. data set $\mathcal{D} \subseteq \mathcal{X} \times \mathcal{Y}$, an integer k; (32)
output: "yes" iff there is a ranking $\gg$ OT-comp. with $\mathcal{D}$ s.t. the corresponding language $\mathcal{L}(\gg)$ has cardinality at most k;
size: $\max\{|\mathcal{C}|, |\mathcal{X}|, width(\mathcal{X})\}$.

The APpbm (32) is clearly in $\mathcal{NP}$, namely it admits a verification algorithm. The following claim 5 together with the NP-completeness of PTAPpbm, entails that the APpbm is NP-complete too, thus completing the proof of claim 2.

Claim 5 *The PTAPpbm (17) can be reduced to the APpbm (32).*

Proof. Given an instance $(\mathbf{A}, \mathcal{F}, k)$ of the PTAPpbm (17), construct the corresponding instance $((\mathcal{X}, \mathcal{Y}, Gen, \mathcal{C}), \mathcal{D}, K)$ of the APpbm (32) as follows. Let m and n be the number of rows and of columns of the comparative tableau $\mathbf{A}$; let ℓ be the cardinality of the set $\mathcal{F}$; let $d = \ell(n - \ell)$. Define the threshold K as in (33).

$$K = m + k + d \tag{33}$$

Define the sets $\mathcal{X}$ and $\mathcal{Y}$ of underlying and surface forms as in (34).

$$
\begin{aligned}
\mathcal{X} &= \underbrace{\{x_1, \ldots, x_m\}}_{\mathcal{X}_1} \cup \underbrace{\{x'_1, \ldots, x'_d\}}_{\mathcal{X}_2} \cup \underbrace{\{x''_1, \ldots, x''_d\}}_{\mathcal{X}_3} \\
\mathcal{Y} &= \underbrace{\begin{Bmatrix} y_1, \ldots, y_m \\ z_1, \ldots, z_m \end{Bmatrix}}_{\mathcal{Y}_1} \cup \underbrace{\begin{Bmatrix} u_1, \ldots, u_d \\ v_1, \ldots, v_d \end{Bmatrix}}_{\mathcal{Y}_2} \cup \underbrace{\begin{Bmatrix} u_1, \ldots, u_d \\ w_1, \ldots, w_d \end{Bmatrix}}_{\mathcal{Y}_3}
\end{aligned} \tag{34}
$$

[8]The generous dependence of the size of the APpbm (11) on $|\mathcal{X}|$ and $width(\mathcal{X})$ provides us with sufficient time to trivially compute the language $\mathcal{L}(\gg)$.

Define the generating function Gen as in (35).

$$\begin{aligned} Gen(x_i) &= \{y_i,\, z_i\} \subseteq \mathcal{Y}_1 \quad \text{for } x_i \in \mathcal{X}_1 \\ Gen(x'_i) &= \{u_i,\, v_i\} \subseteq \mathcal{Y}_2 \quad \text{for } x'_i \in \mathcal{X}_2 \\ Gen(x''_i) &= \{u_i,\, w_i\} \subseteq \mathcal{Y}_3 \quad \text{for } x''_i \in \mathcal{X}_3 \end{aligned} \tag{35}$$

Define the data set $\mathcal{D}$ as in (36).

$$\mathcal{D} = \{(x_1, y_1), \ldots, (x_m, y_m)\} \tag{36}$$

Let the constraint set $\mathcal{C}$ contain a total of n constraints $C_1, \ldots, C_n$; let C_h be a faithfulness constraint iff $h \in \mathcal{F}$, and a markedness constraint otherwise. Since, $Gen(\mathcal{X}_i) \subseteq \mathcal{Y}_i$, constraints need only be defined on $\mathcal{X}_i \times \mathcal{Y}_j$ with $i = j$. The set $\mathcal{X}_1$ contains m underlying forms $x_1, \ldots x_m$, one for every row of the given comparative tableau $\mathbf{A}$. Each of these underlying forms x_i comes with the two candidates y_i and z_i. The data set $\mathcal{D}$ in (36) is a subset of $\mathcal{X}_1 \times \mathcal{Y}_1$. Define the constraints $C_1, \ldots, C_n$ over $\mathcal{X}_1 \times \mathcal{Y}_1$ as in (37). This definition ensures that $\mathbf{A}$ is the comparative tableau corresponding to $\mathcal{D}$, so that (40) holds for any ranking.

$$\gg \text{ is OT-comp. with } \mathbf{A} \text{ iff } \gg \text{ is OT-comp. with } \mathcal{D} \tag{40}$$

The set $\mathcal{X}_2$ contains a total of $d = \ell(n - \ell)$ underlying forms $x'_1, \ldots, x'_2$, one for every pair of a faithfulness constraint and a markedness constraint. Pair up (in some arbitrary but fixed way) each of these underlying forms with a unique pair of a faithfulness constraint and a markedness constraint. Thus, I can speak of "the" markedness constraint and "the" faithfulness constraint "corresponding" to a given underlying form $x'_i \in \mathcal{X}_2$. Each of these underlying forms x'_i comes with two candidates u_i and v_i. Define the constraints $C_1, \ldots, C_n$ over $\mathcal{X}_2 \times \mathcal{Y}_2$ as in (38). This definition ensures that the grammar $\mathsf{OT}_{\gg}$ corresponding to an arbitrary ranking $\gg$ maps x'_i to v_i rather than to u_i iff the faithfulness constraint corresponding to the underlying form x'_i is $\gg$-ranked above the markedness constraint corresponding to x'_i. Since $\mu_{\text{PT}}(\gg)$ is defined in (13) as the total number of pairs of a faithfulness and a markedness constraint such that the former is ranked above the latter, then condition (41) holds for any ranking.

$$\mu_{\text{PT}}(\gg) = |\{x'_i \in \mathcal{X}_2 \,|\, \mathsf{OT}_{\gg}(x'_i) = v_i\}| \tag{41}$$

Finally, define the constraints $C_1, \ldots, C_n$ over $\mathcal{X}_3 \times \mathcal{Y}_3$ as in (38). This definition ensures that the forms $u_1, \ldots, u_d$ are *unmarked* — as the forms [ta] and [rat] in the typology in (1). Thus, they belong to the language corresponding to any ranking $\gg$, as stated in (42).

$$\{u_1, \ldots, u_d\} \subseteq \mathcal{L}(\gg) \tag{42}$$

Assume that the instance $(\mathbf{A}, \mathcal{F}, k)$ of the PTAPpbm admits a positive answer. Thus, there exists a ranking $\gg$ OT-compatible with the comparative tableau $\mathbf{A}$ such that $\mu_{\text{PT}}(\gg) \leq k$. Since $\gg$ is OT-compatible with $\mathbf{A}$, then $\gg$ is OT-compatible with $\mathcal{D}$, by (40). Furthermore, the language $\mathcal{L}(\gg)$ corresponding to the ranking $\gg$ contains at most $K = m + k + d$ surface forms, namely: the m surface forms $y_1, \ldots, y_m \in \mathcal{Y}_1$, because $\gg$ is OT-compatible with $\mathcal{D}$; the d surface forms $u_1, \ldots, u_d$, by (42); and at most k of the surface forms $v_1, \ldots, v_d$, by (41) and the hypothesis that $\mu_{\text{PT}}(\gg) \leq k$. Thus, $\gg$ is a solution of the instance $((\mathcal{X}, \mathcal{Y}, Gen, \mathcal{C}), \mathcal{D}, K)$ of the APpbm (32) constructed in (33)-(39). The same reasoning shows that the vice versa holds too. ■

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$$\begin{aligned} C_h(x_i, y_i) < C_h(x_i, z_i) &\iff \text{the } k\text{th entry in the } i\text{th row of } \mathbf{A} \text{ is a W} \\ C_h(x_i, y_i) = C_h(x_i, z_i) &\iff \text{the } k\text{th entry in the } i\text{th row of } \mathbf{A} \text{ is a E} \\ C_h(x_i, y_i) > C_h(x_i, z_i) &\iff \text{the } k\text{th entry in the } i\text{th row of } \mathbf{A} \text{ is a L} \end{aligned} \tag{37}$$

$$\begin{aligned} C_h(x'_i, v_i) &< C_h(x'_i, u_i) && \text{if } C_h \text{ is the faithfulness constraint corresponding to } x'_i \\ C_h(x'_i, v_i) &> C_h(x'_i, u_i) && \text{if } C_h \text{ is the markedness constraint corresponding to } x'_i \\ C_h(x'_i, v_i) &= C_h(x'_i, u_i) && \text{otherwise} \end{aligned} \tag{38}$$

$$C_h(x'_i, u_i) \leq C_h(x'_i, w_i) \quad \text{for every constraint } C_h \tag{39}$$

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