Appendix: Proof of Lemma 2.2.1

Anonymous ACL submission

Lemma 1. *Let $P \in \mathcal{P}^1_{addsub}$ be a arithmetic word problem with n variables ($|\mathbb{V}_P| = n$), then the followings are true:*

1. *The number of possible applications of part whole formula to the problem P, $N_{partwhole}$ is $(n+1)2^{n-2}+1$.*
2. *The number of possible applications of change formula to the problem P, N_{change} is $3^{n-3}(2n^2+6n+1)-2n+1$.*
3. *The number of possible applications of comparison formula to the problem P, $N_{comparison}$ is $3(n-1)(n-2)$.*
4. *The number of all possible applications to the problem P is $N_{partwhole} + N_{change} + N_{comparison}$.*

Proof. Note that for any application, the set of associated variables must contain the unknown variable x.

1. The applications of the *part whole* formula can be divided into two disjoint cases:

   Case I In this case, the unknown x is assigned to the slot *whole*. The set *parts* then must contain at least 2 elements from $\mathbb{V}_P \setminus \{x\}$. The Numbers of such sets are $2^{n-1} - \binom{n-1}{1} - \binom{n-1}{0} = 2^{n-1} - n$. Hence, the number of *part whole* applications where the unknown variable plays the role of the whole is $2^{n-1} - n$.
   Case II In this case, the unknown x is a member of the set *parts*. The slot *whole* then can be assigned to any of the remaining $n-1$ variables in $n-1$ ways. For each assignment of a variable to the slot *whole*, the rest of the set *parts* can be filled with a set $s \subseteq \mathbb{V}_P \setminus \{x, whole\}$ with size at least 1. Number of such set s is $2^{n-2}-1$. Thus, the number of such applications are $(n-1) * (2^{n-2}-1)$.

   Any application of the *part-whole* concept must either fall in case 1 or case 2 but not both. Thus the number of total *part-whole* applications to a problem $P \in \mathcal{P}^1_{addsub}$ with n variables is $2^{n-1} - n + (n-1) * (2^{n-2}-1)$ = $(n+1)2^{n-2}+1$.

2. First see that, the number of ways in which the *gains*, *losses* slots can be filled with a set S of n variables, such that $gains \bigcup losses \subseteq S$, $gains \bigcap losses = \phi$ and $gains \bigcup losses \neq \phi$ is $3^n - 1$. This is because for each variable in S we have 3 options. We can put it in *gains*, or in *losses* or we can ignore it creating 3^n possibilities. However, the union of *gains* and *losses* should not be empty so want to ignore the case where all the variables in S are ignored.

   All the applications of *change* concepts can be divided into two sets:

   Case I: In this this case, the start is not missing. Let us say $T(n)$ denote the number of ways in which a *change* concept can be instantiated by the n variables in $\mathbb{V}_P$ without the restriction that the associated variables must contain an *unknown* and where the start is not missing. Note that, $T(n) = n(n-1)(3^{n-2}-1)$. Since, the $start$ and the end slots can be filled in $n(n-1)$ ways and for each such choice the *gains* and the *losses* slots can be filled with the remaining $n-2$ variables in $3^{n-2}-1$ ways. Then the number of valid *change* applications in this case is equal to $T(n) - T(n-1)$ i.e. the number of instantiations with or without the *unknown* minus the number of instantiations without the *unknown*.

Case II: In this case, the start is missing. Let us say $T'(n)$ denote the number of ways in which a *change* concept can be instantiated by the n variables in $\mathbb{V}_P$ without the restriction that the associated variables must contain an *unknown* and where the start is missing. Note that, $T'(n) = n(3^{n-1} - 1)$. Following the similar argument as above, the number of valid *change* applications in this case is equal to $T'(n) - T'(n-1)$.

Thus the total number of *change* applications is equal to $T(n) + T'(n) - T(n-1) - T'(n-1)$. After simplifying this we get the desired result.

3. The unknown x can be assigned to any of the three slots *large, small, differenece* in 3 ways. For each such choice for the *unknown* the remaining one of two slots can be filled in $n-1$ ways and for each assignment of the unknown and the one of the two slots, the other can be filled in $n-2$ ways.

4. This follows as we currently consider only three applications and applications of different formulas are different from each other.

□