A Proofs

A.1 Support inverse reverses k-hyponymy

Theorem 1. *For two density matrices A and B, k-hyponymy is reversed by support inverse when $rank(A) = rank(B)$:*

$$A \sqsubseteq_k B \iff \neg_{supp} B \sqsubseteq_k \neg_{supp} A \tag{21}$$

Proof. From (Baksalary et al., 1989), $\neg_{supp}$ reverses Löwner order when $rank(\mathsf{A}) = rank(\mathsf{B})$:

$$\mathsf{A} \sqsubseteq \mathsf{B} \iff \neg_{supp}\mathsf{B} \sqsubseteq \neg_{supp}\mathsf{A} \tag{22}$$

Thus, letting "≥ 0" denote the operator is positive:

$$\begin{aligned}
\mathsf{A} \sqsubseteq_k \mathsf{B} &\iff \mathsf{B} - k\mathsf{A} \geq 0 && (23)\\
&\iff (k\mathsf{A})^{-1} - \mathsf{B}^{-1} \geq 0 && (24)\\
&\iff \frac{1}{k}\mathsf{A}^{-1} - \mathsf{B}^{-1} \geq 0 && (25)\\
&\iff \mathsf{A}^{-1} - k\mathsf{B}^{-1} \geq 0 && (26)\\
&\iff \mathsf{B}^{-1} \sqsubseteq_k \mathsf{A}^{-1} && (27)
\end{aligned}$$

using Equations 5 and 22 from Equation 23 to 24. □

Corollary 1. *For two invertible density matrices A and B, k-hyponymy is reversed by matrix inverse:*

$$A \sqsubseteq_k B \iff B^{-1} \sqsubseteq_k A^{-1} \tag{28}$$

A.2 Matrix inverse reverses k_{BA} in same basis case

Theorem 2. *For two density matrices A and B with the same eigenbasis, k_{BA} is reversed by matrix inverse:*

$$k_{BA}(B^{-1}, A^{-1}) = k_{BA}(A, B) \tag{29}$$

Proof.

$$\begin{aligned}
k_{\mathsf{BA}}(\mathsf{B}^{-1}, \mathsf{A}^{-1}) &= \frac{\sum_i \lambda^i_{\mathsf{A}^{-1}} - \lambda^i_{\mathsf{B}^{-1}}}{\sum_i \left|\lambda^i_{\mathsf{A}^{-1}} - \lambda^i_{\mathsf{B}^{-1}}\right|} && (30)\\
&= \frac{\sum_i \frac{1}{\lambda^i_{\mathsf{A}}} - \frac{1}{\lambda^i_{\mathsf{B}}}}{\sum_i \left|\frac{1}{\lambda^i_{\mathsf{A}}} - \frac{1}{\lambda^i_{\mathsf{B}}}\right|} && (31)\\
&= \frac{\sum_i \lambda^i_{\mathsf{B}} - \lambda^i_{\mathsf{A}}}{\sum_i \left|\lambda^i_{\mathsf{B}} - \lambda^i_{\mathsf{A}}\right|} && (32)\\
&= k_{\mathsf{BA}}(\mathsf{A}, \mathsf{B}) && (33)
\end{aligned}$$

using Equation 13 from Equation 30 to 31. □

A.3 Composing with $\neg_{sub}$ or $\neg_{inv}$ gives maximally mixed support

Theorem 3. *When composing a density matrix X with $\neg_{supp} X$ via* spider*,* fuzz*, or* phaser*, the resulting density matrix has the desired property of being a maximally mixed state on the support with zeroes on the kernel.*

Proof. $\neg_{supp}\mathsf{X}$ and X have the same eigenbasis. From Equation 13, all nonzero eigenvalues of $\neg_{supp}\mathsf{X}$ are multiplicative inverses of the corresponding eigenvalue of X.

We use definitions of *spider*, *fuzz*, and *phaser* from Equations 1, 2, and 3. The summation indices are over eigenvectors with nonzero eigenvalue.

$$\begin{aligned}
&spider(\mathsf{X}, \neg_{supp}\mathsf{X}) && (34)\\
&= U_s(\mathsf{X} \otimes \neg_{supp}\mathsf{X})U_s^\dagger && (35)\\
&= \Big(\sum_i |i\rangle\langle ii|\Big)(\mathsf{X} \otimes \neg_{supp}\mathsf{X})\Big(\sum_j |jj\rangle\langle j|\Big) && (36)\\
&= \sum_i |i\rangle\langle ii| \Big(\big(\lambda|i\rangle\langle i|\big) \otimes \big(\frac{1}{\lambda_i}|i\rangle\langle i|\big)\Big)|ii\rangle\langle i| && (37)\\
&= \sum_i |i\rangle\langle i| && (38)\\
&= \mathbb{I}_{supp} && (39)
\end{aligned}$$

$$\begin{aligned}
fuzz(\mathsf{X}, \neg_{supp}\mathsf{X}) &= \sum_i x_i P_i \circ \mathsf{X} \circ P_i && (40)\\
&= \sum_i \frac{1}{\lambda_i} P_i \Big(\sum_j \lambda_i P_i\Big) P_i && (41)\\
&= \sum_i P_i && (42)\\
&= \mathbb{I}_{supp} && (43)
\end{aligned}$$

$$\begin{aligned}
&phaser(\mathsf{X}, \neg_{supp}\mathsf{X}) && (44)\\
&= \Big(\sum_i x_i P_i\Big) \circ \mathsf{X} \circ \Big(\sum_i x_i P_i\Big) && (45)\\
&= \Big(\sum_i {\lambda_i}^{-\frac{1}{2}} P_i\Big)\Big(\sum_j \lambda_j P_j\Big)\Big(\sum_k {\lambda_k}^{-\frac{1}{2}} P_k\Big) && (46)\\
&= \sum_i P_i && (47)\\
&= \mathbb{I}_{supp} && (48)
\end{aligned}$$

□

Corollary 2. *When composing a density matrix X with $\neg_{inv} X$ via* spider, fuzz, *or* phaser, *the resulting density matrix has the desired property of being a maximally mixed state on the support with zeroes on the kernel.*