

4-VALUED REASONING WITH STRATIFIED BILATTICES

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ABSTRACT

Since [5], 4-valued logic is known to be a useful tool to capture the human reasoning: it is paraconsistent, can treat incompleteness and inconsistency of information etc. In this paper, I propose a 4-valued reasoning system with stratified bilattices of [12]. It inherits desirable formal properties of 4-valued logic, and further realizes a certain kind of default reasoning and truth maintenance system with a simple, lucid LK-style calculus without esoteric, exotic 4-valued operations in [6, 7, 2, 3] etc.

1 Preliminaries

The logical method provides a basic tool to capture the human knowledge and human reasoning. But the traditional 2-valued logic has certain defects in formulating them. I.e., it cannot treat the incomplete or contradictory information which often appears in the human epistemic state. Furthermore, the 2-valued logic is not paraconsistent, i.e., from the contradiction, all statements are derived.

In order to avoid such faults, [5, 1] propose the 4-valued logic. It is based on a very simple idea that the traditional two truth values 'true' and 'false' are treated independently so that the following four truth values are assumed in the 4-valued logic¹:

- (1) i) T (true): A is true.
ii) F (false): A is false.
iii) B (both): A is both true and false.
iv) N (none): A is neither true nor false.

The new truth values B and N represent the contradictory and the incomplete information respectively, i.e. the knowledge states that the traditional 2-valued logic cannot treat because it assumes the following relations between 'true' and 'false'

- (2) A is true $\Leftrightarrow A$ is not false,
 A is false $\Leftrightarrow A$ is not true

so that B and N are impossible at all.

Although the introduction of B and N enables the 4-valued logic to treat the contradictory and the incomplete knowledge state, it is also true that such states tend to change to the contradiction- and incompleteness-free states, i.e. to T or F. In this sense, T and F in the traditional 2-valued logic are the final ideal states which the human knowledge state with contradiction and incompleteness aims at. In order to treat these tendencies, several logical systems have been developed which are classified to the following two types:

- (3) i) $N \rightarrow T$ or F : default logic, circumscription, negation as failure etc.
ii) $B \rightarrow T$ or F : truth maintenance system etc.

In this paper, I present a 4-valued logical system called the situational 4-valued logic S4Val. S4Val can describe the knowledge state and the reasoning with contradictory and incomplete information. Further it can formulate – not all but – some interesting types of the default logic and the truth maintenance system.

In the following, I present the language, the semantics, and the proof theory of S4Val, and then the above-mentioned non-monotonic reasoning.

¹The idea stems from [11].

2 Language of S4Val

The language of S4Val is the same as the ordinary first-order logic, and its formulas are expressed by $speak(h, e)$, $(A \wedge B) \supset C$, $\forall x(P(x) \vee \exists y\neg Q(x, y))$ etc. Var , $Const$, $Func$, $Term$, $Pred$, LC , Q , $Form$ denote the set of variables, constants, function symbols, terms, predicate symbols, logical connectives, quantifiers, and formulas of S4Val, respectively. $Func^n$, $Pred^n$ represent the set of n -ary function symbols and predicate symbols, respectively.

$A, B, C \in Form$, and $t, u \in Term$. Then $A[B]$ means that B may occur in A , and $A[B/C]$ represents the result of uniformly substituting B for C in A . $A[x]$ and $t[x]$ mean that x may appear free in A and t . $A[u/x]$ and $t[u/x]$ represent the result of uniformly substituting u for the free occurrences of x in A and t .

The number of logical connectives and quantifiers in a formula A is called the length of A , and denoted by $Len(A)$.

3 Semantics of S4Val

3.1 Model of S4Val

Definition 1 A model M of S4Val is defined as follows:

$$M = \langle D, \pi, f, v \rangle$$

D : non-empty set of individuals (domain)

$$\pi : \begin{cases} \pi^+ : Pred^n \rightarrow Pow(D^n) \\ \pi^- : Pred^n \rightarrow Pow(D^n) \end{cases}$$

$$f : Func^n \rightarrow (D^n \rightarrow D) \quad (n \geq 0)$$

$$v : Var \rightarrow D \quad (\text{variable assignment})$$

Definition 2 Let v be a variable assignment, and $\langle x, d_0 \rangle \in v$. Then $(v \setminus \{\langle x, d_0 \rangle\}) \cup \{\langle x, d \rangle\}$ (in notation: v_d^x) is called (x, d) -variant of v .

Here, function symbols and variables are interpreted as in the ordinary first-order logic. But the interpretation function π of predicate symbols are splitted into π^+ and π^- which give a predicate symbol its positive and negative domain respectively, i.e. the domains which return T and F respectively if the n -tuple of individuals which the arguments of the predicate denote belongs to the respective domains.

Based on Definition 1, terms and formulas are interpreted with respect to a model as follows:

Definition 3 Let $M = \langle D, \pi, f, v \rangle$ be a model of S4Val. Then the function symbols, predicate symbols, variables, and terms are interpreted by the function $D(\pi, f, v, \cdot)$ as follows:

- i) $D(\pi, f, v, \cdot) \upharpoonright Func \doteq f$.
- ii) $D(\pi^+, f, v, \cdot) \upharpoonright Pred \doteq \pi^+$.
- iii) $D(\pi^-, f, v, \cdot) \upharpoonright Pred \doteq \pi^-$.
- iii) $D(\pi, f, v, \cdot) \upharpoonright Var \doteq v$.
- iv) Let $t_1, \dots, t_n \in Term$, and $f_i^n \in Func^n$. Then $D(\pi, f, v, f_i^n(t_1, \dots, t_n)) \doteq f(f_i^n)(D(\pi, f, v, t_1), \dots, D(\pi, f, v, t_n)) \in D$.

3.2 Combinatorial Interpretation of Formulas

As to the formulas, I first present the interpretation in the traditional 2-valued logical style where the truth values 'true' and 'false' are treated independently:

Definition 4 $t_1, \dots, t_n \in Term$; $pred_i^n \in Pred^n$; $A, B \in Form$. Then the formulas of S4Val are interpreted by the support relations \models^+ and \models^- with respect to the model $M = \langle D, \pi, f, v \rangle$ as follows:

- i) a) $D, \pi, f, v \models^+ pred_i^n(t_1, \dots, t_n) \Leftrightarrow \langle D(\pi, f, v, t_1), \dots, D(\pi, f, v, t_n) \rangle \in \pi^+(pred_i^n)$.
- b) $D, \pi, f, v \models^- pred_i^n(t_1, \dots, t_n) \Leftrightarrow \langle D(\pi, f, v, t_1), \dots, D(\pi, f, v, t_n) \rangle \in \pi^-(pred_i^n)$.
- ii) a) $D, \pi, f, v \models^+ \neg A \Leftrightarrow D, \pi, f, v \models^- A$.
- b) $D, \pi, f, v \models^- \neg A \Leftrightarrow D, \pi, f, v \models^+ A$.

- iii) a) $D, \pi, f, v \models^+ A \wedge B \Leftrightarrow D, \pi, f, v \models^+ A$ and $D, \pi, f, v \models^+ B$.
 b) $D, \pi, f, v \models^- A \wedge B \Leftrightarrow D, \pi, f, v \models^- A$ or $D, \pi, f, v \models^- B$.
- iv) a) $D, \pi, f, v \models^+ A \vee B \Leftrightarrow D, \pi, f, v \models^+ A$ or $D, \pi, f, v \models^+ B$.
 b) $D, \pi, f, v \models^- A \vee B \Leftrightarrow D, \pi, f, v \models^- A$ and $D, \pi, f, v \models^- B$.
- v) a) $D, \pi, f, v \models^+ A \supset B \Leftrightarrow D, \pi, f, v \models^- A$ or $D, \pi, f, v \models^+ B$.
 b) $D, \pi, f, v \models^- A \supset B \Leftrightarrow D, \pi, f, v \models^+ A$ and $D, \pi, f, v \models^- B$.
- vi) a) $D, \pi, f, v \models^+ \forall x A \Leftrightarrow$ for all $d \in D$ $D, \pi, f, v_d^x \models^+ A$.
 b) $D, \pi, f, v \models^- \forall x A \Leftrightarrow$ for some $d \in D$ $D, \pi, f, v_d^x \models^- A$.
- vii) a) $D, \pi, f, v \models^+ \exists x A \Leftrightarrow$ for some $d \in D$ $D, \pi, f, v_d^x \models^+ A$.
 b) $D, \pi, f, v \models^- \exists x A \Leftrightarrow$ for all $d \in D$ $D, \pi, f, v_d^x \models^- A$.

Here, $D, \pi, f, v \models^+ A$ and $D, \pi, f, v \models^- A$ mean that A is (at least) true and false respectively. So, according to (1), the four truth values are expressed by combining truth and falsity as follows:

Definition 5 $A \in Form$, and $M = \langle D, \pi, f, v \rangle$ be a model of S4Val. Then the truth value of A with respect to M is defined as follows:

- i) The truth value of A with respect to M is T: $M \models^+ A$, and $M \not\models^- A$.
- ii) The truth value of A with respect to M is F: $M \not\models^+ A$, and $M \models^- A$.
- iii) The truth value of A with respect to M is B: $M \models^+ A$, and $M \models^- A$.
- iv) The truth value of A with respect to M is N: $M \not\models^+ A$, and $M \not\models^- A$.

Here, $M \not\models^+ A$, and $M \not\models^- A$ represent the negation of $M \models^+ A$, and $M \models^- A$ respectively. But in S4Val, there are no relations as in (2) between $M \models^+ A$, $M \models^- A$ and their negated forms, which enables S4Val to express the four truth values.

3.3 Direct Interpretation of Formulas

Now, the truth values can be directly assigned to formulas of S4Val too. For this, we introduce the bilattice FOUR of [12]. The bilattice itself is defined as follows [12, p.270]:

Definition 6 A bilattice is a sextuple $\langle B, \wedge, \vee, \otimes, \oplus, \sim \rangle$ such that

- i) $\langle B, \wedge, \vee \rangle$, $\langle B, \otimes, \oplus \rangle$ are both complete lattices, and
- ii) For any $a \in B$, $\sim \sim a = a$.
- iii) \sim is a lattice homomorphism from $\langle B, \wedge, \vee \rangle$ to $\langle B, \vee, \wedge \rangle$ and from $\langle B, \otimes, \oplus \rangle$ to itself. I.e., for arbitrary $a, b \in B$,
 - a) $\sim (a \wedge b) = \sim a \vee \sim b$, and
 $\sim (a \vee b) = \sim a \wedge \sim b$
 - b) $\sim (a \otimes b) = \sim a \otimes \sim b$, and
 $\sim (a \oplus b) = \sim a \oplus \sim b$

FOUR is depicted as the following structure:

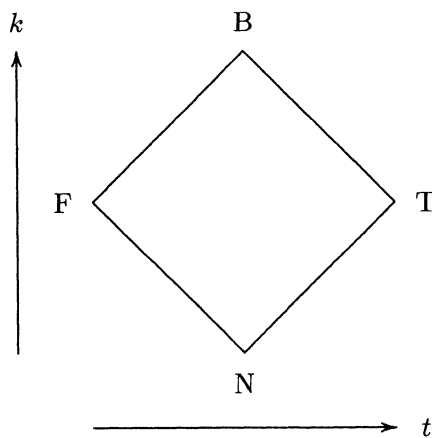


Figure 1: FOUR

Here, $B = \{T, F, B, N\}$. FOUR constitutes two lattices towards the t - and k -axis. (t and k mean ‘truth’ and ‘knowledge’ respectively.) They are called the t - and k -lattice of FOUR respectively. The partial orders in respective axes are represented by \leq_t and \leq_k . \wedge , \otimes and \vee , \oplus are infimum and supremum operation with respect to \leq_t and \leq_k . And \sim in FOUR is a 180° rotation around the B-N-axis so that $\sim B = B$, $\sim N = N$, $\sim F = T$, and $\sim T = F$.

Now, the direct interpretation of formulas which corresponds to Definition 5 is defined as follows:

Definition 7 $A \in Form$, and $M = \langle D, \pi, f, v \rangle$ be a model of S4Val. Then the truth value $D(\pi, f, v, A)$ of A with respect to M is defined as follows:

- i) a) $D(\pi, f, v, pred_i^n(t_1, \dots, t_n)) = T \Leftrightarrow \langle D(\pi, f, v, t_1), \dots, D(\pi, f, v, t_n) \rangle \in \pi^+(pred_i^n)$,
and $\langle D(\pi, f, v, t_1), \dots, D(\pi, f, v, t_n) \rangle \notin \pi^-(pred_i^n)$.
- b) $D(\pi, f, v, pred_i^n(t_1, \dots, t_n)) = F \Leftrightarrow \langle D(\pi, f, v, t_1), \dots, D(\pi, f, v, t_n) \rangle \notin \pi^+(pred_i^n)$,
and $\langle D(\pi, f, v, t_1), \dots, D(\pi, f, v, t_n) \rangle \in \pi^-(pred_i^n)$.
- c) $D(\pi, f, v, pred_i^n(t_1, \dots, t_n)) = B \Leftrightarrow \langle D(\pi, f, v, t_1), \dots, D(\pi, f, v, t_n) \rangle \in \pi^+(pred_i^n)$,
and $\langle D(\pi, f, v, t_1), \dots, D(\pi, f, v, t_n) \rangle \in \pi^-(pred_i^n)$.
- d) $D(\pi, f, v, pred_i^n(t_1, \dots, t_n)) = N \Leftrightarrow \langle D(\pi, f, v, t_1), \dots, D(\pi, f, v, t_n) \rangle \notin \pi^+(pred_i^n)$,
and $\langle D(\pi, f, v, t_1), \dots, D(\pi, f, v, t_n) \rangle \notin \pi^-(pred_i^n)$.
- ii) $D(\pi, f, v, \neg A) = \sim D(\pi, f, v, A)$.
- iii) $D(\pi, f, v, A \wedge B) = D(\pi, f, v, A) \wedge D(\pi, f, v, B)$.
- iv) $D(\pi, f, v, A \vee B) = D(\pi, f, v, A) \vee D(\pi, f, v, B)$.
- v) $D(\pi, f, v, A \supset B) = \sim D(\pi, f, v, A) \vee D(\pi, f, v, B)$.
- vi) $D(\pi, f, v, \forall x A) = \bigwedge_{d \in D} D(\pi, f, v_d^x, A)$.
- vii) $D(\pi, f, v, \exists x A) = \bigvee_{d \in D} D(\pi, f, v_d^x, A)$.

The equivalence of both interpretations Definition 5 and Definition 7 is secured by the following theorem:

Theorem 1

- i) $D(\pi, f, v, A) = T \Leftrightarrow D, \pi, f, v \models^+ A$, and $D, \pi, f, v \not\models^- A$.
- ii) $D(\pi, f, v, A) = F \Leftrightarrow D, \pi, f, v \not\models^+ A$, and $D, \pi, f, v \models^- A$.
- iii) $D(\pi, f, v, A) = B \Leftrightarrow D, \pi, f, v \models^+ A$, and $D, \pi, f, v \models^- A$.
- iv) $D(\pi, f, v, A) = N \Leftrightarrow D, \pi, f, v \not\models^+ A$, and $D, \pi, f, v \not\models^- A$.

Proof We prove this by the induction on the length of formulas.

I) If $Len(A) = 0$, it obviously holds by Definition 4i) and Definition 7i).

II) Assume that the theorem holds with $Len(B), Len(C) \leq k$. Then the theorem also holds with $Len(A) = k + 1$ as follows:

i) $A \doteq \neg B$:

a) $D(\pi, f, v, \neg B) = T$: Then $\sim D(\pi, f, v, B) = T$ by Definition 7ii). So $D(\pi, f, v, B) = F$. By the induction hypothesis, $D, \pi, f, v \not\models^+ B$, and $D, \pi, f, v \models^- B$. So, by Definition 4ii) $D, \pi, f, v \models^+ \neg B$, and $D, \pi, f, v \not\models^- \neg B$. The converse also holds.

b,c,d) $D(\pi, f, v, \neg B) = F, B$, and N : Omitted.

ii) $A \doteq B \wedge C$:

a) $D(\pi, f, v, B \wedge C) = T$: Then $D(\pi, f, v, B) \wedge D(\pi, f, v, C) = T$ by Definition 7iii). Then $D(\pi, f, v, B) = T$, and $D(\pi, f, v, C) = T$. By the induction hypothesis, $D, \pi, f, v \models^+ B$; $D, \pi, f, v \not\models^- B$, and $D, \pi, f, v \models^+ C$; $D, \pi, f, v \not\models^- C$. Then by Definition 4iii), $D, \pi, f, v \models^+ B \wedge C$, and $D, \pi, f, v \not\models^- B \wedge C$. The converse also holds.

b,c,d) $D(\pi, f, v, B \wedge C) = F, B$, and N : Omitted.

iii),iv) $A \doteq B \vee C, B \supset C$: Omitted.

v) $A \doteq \forall x B$:

a) $D(\pi, f, v, \forall x B) = T$: Then $\bigwedge_{d \in D} D(\pi, f, v_d^x, B) = T$ by Definition 7vi). Then, for all $d \in D$, $D(\pi, f, v_d^x, B) = T$. By the induction hypothesis, $D, \pi, f, v_d^x \models^+ B$, and $D, \pi, f, v_d^x \not\models^- B$ for all $d \in D$. Then by Definition 4vi), $D, \pi, f, v \models^+ \forall x B$, and $D, \pi, f, v \not\models^- \forall x B$. The converse also holds.

b,c,d) $D(\pi, f, v, \forall x B) = F, B$, and N : Omitted.

vi) $A \doteq \exists x B$: Omitted. □

Further, if we decompose $D(\pi, f, v, A)$ to its truth and falsity factor $D^+(\pi, f, v, A)$ and $D^-(\pi, f, v, A)$ according to the following definition:

Definition 8 $A \in Form$. Then $D^+(\pi, f, v, A)$ and $D^-(\pi, f, v, A)$ are defined as follows:

- i) a) $D^+(\pi, f, v, A) = T \Leftrightarrow D, \pi, f, v \models^+ A$.
- b) $D^+(\pi, f, v, A) = N \Leftrightarrow D, \pi, f, v \not\models^+ A$.
- ii) a) $D^-(\pi, f, v, A) = F \Leftrightarrow D, \pi, f, v \models^- A$.
- b) $D^-(\pi, f, v, A) = N \Leftrightarrow D, \pi, f, v \not\models^- A$.

$D(\pi, f, v, A)$ is also expressed combinatorially by $D^+(\pi, f, v, A)$ and $D^-(\pi, f, v, A)$, as is proved by the following theorem:

Theorem 2 $A \in Form$, and $M = \langle D, \pi, f, v \rangle$ be a model of S4Val. Then

$$D(\pi, f, v, A) = D^+(\pi, f, v, A) \oplus D^-(\pi, f, v, A).$$

Proof i) $D(\pi, f, v, A) = T$, then $M \models^+ A$, and $M \not\models^- A$ by Theorem 1.i). Then by Definition 8, it means that $D^+(\pi, f, v, A) = T$, and $D^-(\pi, f, v, A) = N$. So, $D^+(\pi, f, v, A) \oplus D^-(\pi, f, v, A) = T$.

ii,iii,iv) The case with the truth value F, B, N is proved in the same manner. \square

3.4 Inference Relation of S4Val

Next, we define the semantic inference of S4Val. For this, we first define the interpretation of S4Val.

Definition 9 Let $M = \langle D, \pi, f, v \rangle$ be a model of S4Val. Then

$$I = \langle D, \pi, f \rangle$$

is called an interpretation of S4Val.

Based on Definition 9, the inference relation of S4Val is defined as follows:

Definition 10 Let I be an interpretation of S4Val, $\Gamma \subseteq Form$, $A \in Form$, and $pol \in \{+, -\}$. Then

- i) $I \models^{pol} A \Leftrightarrow$ for an arbitrary variable assignment v : $I, v \models^{pol} A$.
 - ii) $I, v \models^{pol} \Gamma \Leftrightarrow$ for an arbitrary $A \in \Gamma$: $I, v \models^{pol} A$.
 - iii) $I \models^{pol} \Gamma \Leftrightarrow$ for an arbitrary variable assignment v : $I, v \models^{pol} \Gamma$.
- If $\Gamma = \{A\}$, Γ is abbreviated to A .

Definition 11 Let I be an interpretation of S4Val; $\Gamma, \Delta \subseteq Form$. Then the inference relation \models^+ of S4Val is defined as follows:

$$\Gamma \models^+ \Delta \Leftrightarrow \text{for an arbitrary } I: I \models^+ \Gamma \Rightarrow I \models^+ \Delta.$$

\models^+ is often abbreviated to \models . E.g., $I \models A$, $\Gamma \models \Delta$ for $I \models^+ A$, $\Gamma \models^+ \Delta$.

3.5 Some Semantic Properties of S4Val

S4Val with the above-mentioned interpretation has some interesting semantic properties. I.e., it lacks some of the basic properties of the traditional 2-valued logic.

Theorem 3 In ii)-vi) below, let M and A, B be (not arbitrary but) some model and formulas of S4Val respectively.

- i) There is no tautology in S4Val. I.e., there is no formula A such that $\models A$.
- ii) The law of excluded middle does not hold in S4Val. I.e., $M \not\models A \vee \neg A$.
- iii) The law of contradiction does not hold in S4Val. I.e., $M \models A \wedge \neg A$.
- iv) Modus Ponens does not hold in S4Val. I.e., $M \models A$, and $M \models A \supset B \not\Rightarrow M \models B$.
- v) Deduction Theorem does not hold in S4Val. I.e., $\Gamma, A \models B \not\Rightarrow \Gamma \models A \supset B$.
- vi) S4Val is paraconsistent. I.e., $A, \neg A \not\models B$.

Proof

- i) A model which assigns the empty set to every predicate symbol P , i.e. where $\pi^+(P) = \phi$, and $\pi^-(P) = \phi$, assigns the truth value N to every formula.
- ii) Consider a model which assigns N to A .
- iii) Consider a model which assigns B to A .
- iv) Suppose that $M(A) = B$, and $M(B) = N$. Then $M \models A$, and $M \models A \supset B$, but $M \not\models B$.
- v) Suppose that $M(\alpha) = T$ for every element α in Γ , and $M(A) = N$. Further, $A \doteq B$. Then $\Gamma, A \models B$. but $\Gamma \not\models A \supset B$.
- vi) Consider a model which assigns B and N to A and B respectively. \square

N.B. Theorem 3i-vi) are all proved using the truth values B or N, i.e. the truth values which the traditional 2-valued logic lacks.

3.6 Inference Relation with Constraints

For practical purposes, it is often useful to pose some constraints on the inference relation which are classified to the following two types:

- (4) i) axioms.
- ii) inferential rules.

- (5) i) $japanese(taro)$.
- ii) $japanese(\alpha) \models kind(\alpha)$. (α is an arbitrary term. Likewise in the following.)

are examples for the respective types.

If we pose the constraint (5i), then I in Definition 9 is restricted to the interpretation where $I \models japanese(taro)$. And if we pose the constraint (5ii), the above I is restricted so that (5ii) holds.

In the following, \models stands for the inference relation with the above constraints. If we will explicitly express such constraints, we use the notation $Axiom \models_{Inf}$ with $Axiom$ and Inf as the constraints of (4i) and (4ii) respectively.

N.B. from $\models A$, and $A \models B$, it follows that $\models B$. But from $\models A$, and $\models A \supset B$, it does not always follow that $\models B$, as is proved by Theorem 3.iv).

3.7 Bilattice DEFAULT and Non-monotonic Reasoning

[12] proposes a bilattice called DEFAULT which is depicted as Figure 2:

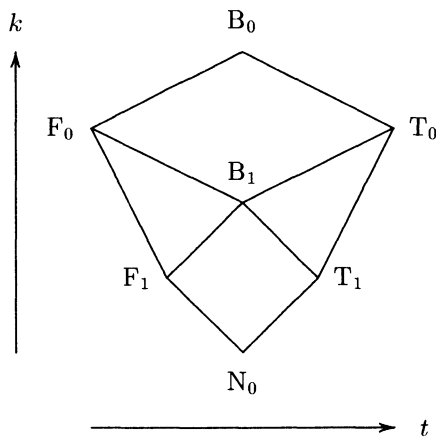


Figure 2: DEFAULT

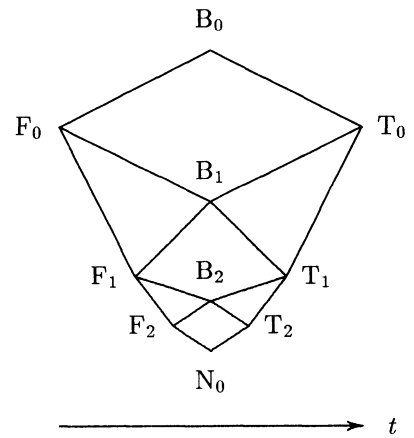


Figure 3: DEFAULT

Here, the truth values T, F, B, N of FOUR are indexed so that DEFAULT consists of 7 elements $B_0, F_0, T_0, B_1, F_1, T_1, N_0$. The elements B_1, F_1, T_1, N_0 are considered as a refinement of N in FOUR. Conversely, FOUR is considered as a special case of DEFAULT with no refinement of N_0 . The intention of DEFAULT is the following: If we assign T_1 to the formula

- (6) $bird(tweety)$,

(6) is true by default. But if we also give (6) F_0 after that, T_1 is overridden by F_0 , and the truth value of (6) is now F_0 , which means that we can capture a certain type of default reasoning by means of DEFAULT.

Furthermore, assume that (6) has the truth value B_1 at the beginning, and then it is given the truth value T_0 . Then it means that DEFAULT can capture a certain kind of truth maintenance system.²

We can generalize DEFAULT to bilattices with further refinements of N_0 . Figure 3 is an example for it. We also call it DEFAULT. As is seen from Figure 3, DEFAULT bilattices constitute a stratification. I.e., a DEFAULT bilattice contains a smaller DEFAULT bilattice until it reaches FOUR. So they belong to the class of the stratified bilattices of [12, p.285] so that the above-mentioned types of non-monotonic reasoning amount to a case of reasoning using stratified bilattices.

Below, I define some terms concerning the stratification.

²As to the default logic and the truth maintenance system, cf. [14, 10] etc.

Definition 12 Let E be a DEFAULT bilattice, and $E^x = \{y \in E \mid x \not\prec y\}$. If E^x is also a DEFAULT (including FOUR), x is called a splitting point of E .

The number $Dgr(E)$ of the splitting points in E is called the degree (of stratification) of E .

$Str(E^x) = Dgr(E) - Dgr(E^x)$ is called the stratum number of E^x . E^x is called the $Str(E^x)$ -th stratum of E .

$S4Val^d$ with the degree d is the collection of $S4Val_s^d$ ($0 \leq s \leq d-1$) with the stratum number s . (d is omitted in obvious cases.)

Here, in Figure 3, $E^{B_0} = E$. So $Str(E) = 0$. Further, $Str(E^{B_1}) = 1$, and $Str(E^{B_2}) = 2$.

Now, the semantic interpretation on FOUR must be altered to the stratum specific one which consists of the introduction of the stratum number. But the change is minimal, and we only need to add the subscript s of the stratum number to the symbols defined for the above stratum-number-free semantics: e.g., M_s , I_s , $S4Val_s$. Further, the collection of M_s ($0 \leq s \leq d-1$) in DEFAULT with the degree d is denoted by $M^d = \langle D, \pi, f, v \rangle^d$ (also without d in obvious cases).

Now, the stratum number s stems from the relativization of π with respect to s in the model $M_s = \langle D, \pi_s, f, v \rangle$ such that

$$(7) \quad \pi : \begin{cases} \pi_s^+ : Pred^n \rightarrow Pow(D^n) \\ \pi_s^- : Pred^n \rightarrow Pow(D^n) \end{cases}$$

D, f, v remain unchanged. So, the correspondence with the stratum number s to the definitions and theorems in 3.1–3.5 hold with π_s instead of π . Likewise, T, F, B are replaced by T_s, F_s, B_s , respectively. But notice that N is replaced by N_0 . I.e., if a formula is N in all strata, it is finally N, i.e. N_0 .

The interpretation with the stratum number gives a formula stratum specific truth values. But now, what is the truth value of a formula with respect to such an interpretation? It is given by the following definition:

Definition 13 The truth value of a formula A with respect to DEFAULT with the degree d is given by $\oplus_{i=0}^{d-1} D(\pi_i, f, v, A)$.

The following two theorems immediately follow from Definition 13.

Theorem 4 The truth value of the formula A with respect to the model $M^d = \langle D, \pi, f, v \rangle^d$ is $D(\pi_s, f, v, A)$ if there is the least stratum number s such that $D(\pi_s, f, v, A)$ is B_s, T_s or F_s , otherwise N_0 .

Proof If there is such $D(\pi_s, f, v, A)$, then $D(\pi_s, f, v, A) \geq_k D(\pi_t, f, v, A)$ for all $t(\geq s)$. Then $\oplus_{i=0}^{d-1} D(\pi_i, f, v, A) = D(\pi_s, f, v, A)$. Otherwise $D(\pi_s, f, v, A) = N_0$ for all t . Then $\oplus_{i=0}^{d-1} D(\pi_i, f, v, A) = N_0$. \square

Theorem 5 The truth value of the formula A with respect to the model $M = \langle D, \pi, f, v \rangle$ is given as follows:

If there is the least stratum number s such that $D^+(\pi_s, f, v, A) = T_s$ or $D^-(\pi_s, f, v, A) = F_s$, the truth value of A is $D^+(\pi_s, f, v, A) \oplus D^-(\pi_s, f, v, A)$. Otherwise it is N_0 .

Proof Immediate from the stratum number specific correspondence of Theorem 1, i.e., $D(\pi_s, f, v, A) = D^+(\pi_s, f, v, A) \oplus D^-(\pi_s, f, v, A)$, and Theorem 4. \square

Theorem 4 and 5 capture the history about (6).

We can further define the semantic constraints with the stratum number s which correspond to (4) and (5) as follows.

- (8) i) axioms with the stratum number s .
- ii) inferential rules with the stratum number s .

Such constraints are represented with the index of the stratum number as follows:

- (9) i) $japanese(taro)_s$.
- ii) $japanese(\alpha) \models_s kind(\alpha)$.

We assume that such constraints are downward-valid, but not vice versa. I.e.,

Definition 14

- i) If $I_s \models_s A$, then $I_t \models_t A$ for all $t \geq s$.
- ii) If $A \models_s B$, then $A \models_t B$ for all $t \geq s$.

In the following, we omit the stratum number s in obvious cases.

4 Proof Theory SLK4 of S4Val

Now, I introduce the proof theory SLK4 of S4Val with stratification. But at first, I give the following definition.

Definition 15 A row of finite (including 0) formulas separated each other by a comma is called a sequence (of formulas). It is also denoted by a Greek capital letter.

Let A_1, \dots, A_m and B_1, \dots, B_n be two sequences. Then

i) $A_1, \dots, A_m \rightarrow_s B_1, \dots, B_n$

is called a sequent with the stratum number s .

i) is equivalent to the formula

ii) $(\neg A_1 \vee \dots \vee \neg A_m \vee B_1 \vee \dots \vee B_n)_s$.

A set of literals is called a clause. It is interpreted disjunctively. And a set of clauses is interpreted conjunctively.

The proof theory $SLK4^d$ of S4Val with the degree d of stratification is the collection of the proof theories $SLK4_s^d$ with the stratum number s ($0 \leq s \leq d-1$). (d is omitted in obvious cases.)

If the formulas A_i, B_j ($1 \leq i \leq m, 1 \leq j \leq n$) in Definition 15ii) are atomic, Definition 15ii) is also represented in the following clausal form:

(10) $\{\neg A_1, \dots, \neg A_m, B_1, \dots, B_n\}_s$.

A sequent in its original, formulaic (and further in its clausal) form are treated equivalently. So for example, i)[t/x] means ii)[t/x] in Definition 15.

Till now, several LK-style 4-valued systems have been proposed. E.g., LK system with specific 4-valued connectives of [13, chap.6], or bisequent system of [8]. But $SLK4_s$ developed here is rather conservative. It is essentially Gentzens LK, only with the following modifications:

- (11) 1) Axioms and inferential rules are relativized with respect to the stratum.
 2) The axiom $A \rightarrow_s A$ is not always assumed.
 3) Instead of 2), several proper axioms are assumed.
 4) Cut is not assumed.
 5) Several proper inferential rules are assumed.

$SLK4_s$ consists of the following axioms and inferential rules:

Definition 16

Proper Axioms $A_1, \dots, A_m; B_1, \dots, B_n \in AForm$. Then the set $Axiom_s$ of the axioms of $SLK4_s$ consists of the sequents in the following form:

$$A_1, \dots, A_m \rightarrow_s B_1, \dots, B_n \quad (m, n \geq 0, m + n \geq 1)$$

Inferential Rules

Uniform Substitution Rule (US)

Let seq be a sequent, $t \in Term$, and t be free for x in seq . Then

$$\frac{seq}{seq[t/x]}_s$$

Structural Rules

WL (Weakning Left)

$$\frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}_s$$

WR (Weakning Right)

$$\frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}_s$$

CL (Contraction Left)

$$\frac{A, A, \Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}_s$$

CR (Contraction Right)

$$\frac{\Gamma \rightarrow \Delta, A, A}{\Gamma \rightarrow \Delta, A}_s$$

EL (Exchange Left)

$$\frac{\Gamma, A, B, \Pi \rightarrow \Delta}{\Gamma, B, A, \Pi \rightarrow \Delta}_s$$

ER (Exchange Right)

$$\frac{\Gamma \rightarrow \Delta, A, B, \Sigma}{\Gamma \rightarrow \Delta, B, A, \Sigma}_s$$

Logical Rules

\neg -L (Negation Left)

$$\frac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta}^s$$

\neg -R (Negation Right)

$$\frac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A}^s$$

\wedge -L1 (Conjunction Left 1)

$$\frac{A, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}^s$$

\wedge -L2 (Conjunction Left 2)

$$\frac{B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}^s$$

\wedge -R (Conjunction Right)

$$\frac{\Gamma \rightarrow \Delta, A \quad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \wedge B}^s$$

\vee -L (Disjunction Left)

$$\frac{A, \Gamma \rightarrow \Delta \quad B, \Gamma \rightarrow \Delta}{A \vee B, \Gamma \rightarrow \Delta}^s$$

\vee -R1 (Disjunction Right 1)

$$\frac{\Gamma \rightarrow \Delta, A}{\Gamma \rightarrow \Delta, A \vee B}^s$$

\vee -R2 (Disjunction Right 2)

$$\frac{\Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \vee B}^s$$

\supset -L (Implication Left)

$$\frac{\Gamma \rightarrow \Delta, A \quad B, \Pi \rightarrow \Sigma}{A \supset B, \Gamma, \Pi \rightarrow \Delta, \Sigma}^s$$

\supset -R (Implication Right)

$$\frac{A, \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \supset B}^s$$

Quantificational Rules

\forall -L (Universal Quantification Left)

$$\frac{A[t/x], \Gamma \rightarrow \Delta}{\forall x A, \Gamma \rightarrow \Delta}^s$$

\forall -R (Universal Quantification Right)

$$\frac{\Gamma \rightarrow \Delta, A[y/x]}{\Gamma \rightarrow \Delta, \forall x A}^s$$

\exists -L (Existential Quantification Left)

$$\frac{A[y/x], \Gamma \rightarrow \Delta}{\exists x A, \Gamma \rightarrow \Delta}^s$$

\exists -R (Existential Quantification Right)

$$\frac{\Gamma \rightarrow \Delta, A[t/x]}{\Gamma \rightarrow \Delta, \exists x A}^s$$

t and y in quantification rules must be free for x in A . Further, in \forall -R and \exists -L, y must not appear in the conclusion.

Proper Inferential Rules

Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be sequences of atomic formulas, and $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ be sequences of formulas. Then the proper inferential rules of SLK4_s consist of the inferential rules in the following form:

$$\frac{\Gamma_1, \alpha_1, \Gamma_2 \rightarrow \Delta_1, \beta_1, \Delta_2}{\Gamma_1, \alpha_2, \Gamma_2 \rightarrow \Delta_1, \beta_2, \Delta_2}^s$$

Here, only the proper axioms and proper inferential rules are stratum number specific. The other inferential rules hold for every stratum number s so that the index s could be omitted. But it is necessary to show in which stratum the inference is executed.

Notice that axioms consist of atomic formulas. In SLK4_s , a formula of the first-order logic is converted to a skolemized prenex conjunctive normal form. And each conjunct is registered as an axiom of SLK4_s . In this regard, SLK4_s is considered to be a restricted system of the first-order theory.

Free variables in sequences are interpreted universally. So Axiom_s restricts the set of admissible interpretations (but not models) to $\mathcal{I} = \{I \mid I \models_s^+ \text{Axiom}_s\}$. Likewise, the Uniform Substitution Rule gets possible due to this universal quantification over the variables.

In SLK4_s ,

- (12) i) the axiom of Gentzen's LK in the form $A \rightarrow_s A$ is not always assumed.

On the other hand,

- ii) we can assume $A \rightarrow_s$ and $\rightarrow_s A$.

(12i) and (12ii) mean that the excluded middle and the law of contradiction do not always hold in SLK4_s .

If Axiom_s contains exactly one of $A \rightarrow_s$ and $\rightarrow_s A$ for all $A \in \text{AForm}$, it would represent a possible world. But in general, Axiom_s with (12i,ii) represents an underdetermined and overdetermined state of information. Further, if Axiom_s contains $\rightarrow_s A, B$, it represents a disjunctive state of information where A or B holds, but it is not known which. So Axiom_s does not represent a possible world, but in general a situation, which is the reason why the name S4Val and SLK4 contain the term "situation".³

³As to the situation theory, cf. [4, 9] etc. Of course, the term "situation" is used here in a wider sense than its usual one because it also includes the contradictory and disjunctive information.

Inferential rules are intended to semantically mean Definition 11 with stratification. I.e., $\frac{seq_1}{seq_2}^s$ means $seq_1 \models_s seq_2$.

In $SLK4_s$, Cut is not assumed from the semantic viewpoint. For this, consider the interpretation where A is B , and every formula in $\Gamma, \Pi, \Delta, \Sigma$ is N in the following Cut:

$$(13) \frac{\Gamma \rightarrow \Delta, A \quad A, \Pi \rightarrow \Sigma}{\Gamma, \Pi \rightarrow \Delta, \Sigma}^s$$

On the other hand, proper inferential rules pose semantic constraints on $S4Val_s$. E.g.,

$$(14) \frac{\Gamma \rightarrow \Delta, \text{japanese}(\alpha), \Sigma}{\Gamma \rightarrow \Delta, \text{kind}(\alpha), \Sigma}^s$$

is the proof-theoretical counterpart of

$$(15) \Gamma \rightarrow \Delta, \text{japanese}(\alpha), \Sigma \models \Gamma \rightarrow \Delta, \text{kind}(\alpha), \Sigma.$$

We also assume the proof-theoretical counterpart of Definition 14. I.e.,

Definition 17

- i) If $A_1, \dots, A_m \rightarrow_s B_1, \dots, B_n$ is an axiom of $SLK4_s$, then $A_1, \dots, A_m \rightarrow_t B_1, \dots, B_n$ for all $t \geq s$.
- ii) If $\frac{\Gamma \rightarrow \Delta}{\Pi \rightarrow \Sigma}^s$, then $\frac{\Gamma \rightarrow \Delta}{\Pi \rightarrow \Sigma}^t$ for all $t \geq s$.

The deductive relation of $SLK4$ is defined as follows:

Definition 18 \mathcal{A}, \mathcal{B} be a (series of) sequent(s). Then

$$\text{Axiom}, \mathcal{A} \vdash_{\text{SLK4}_s, \text{Inf}} \mathcal{B}$$

represents that the sequent (or every sequents in) \mathcal{B} is deduced from \mathcal{A} in $SLK4_s$ with the set *Axiom* of proper axioms and the set *Inf* of proper inferential rules.

In obvious cases, we omit *Axiom*, *Inf*, and s in $\text{Axiom}, \mathcal{A} \vdash_{\text{SLK4}_s, \text{Inf}} \mathcal{B}$, or use s instead of $SLK4_s$.

$SLK4_s$ is sound and complete with respect to $S4Val_s$. I.e.,

Theorem 6 For an arbitrary $A \in \text{Form}$

$$\text{Axiom} \vdash_{s, \text{Inf}} A \Leftrightarrow \text{Axiom} \models_{s, \text{Inf}} A$$

Proof Omitted. □

Based on Theorem 6, the proof theoretical counterpart of Theorem 3 holds. I.e.,

Theorem 7 In ii)-vi) below, let $SLK4_s$ and A, B be (not arbitrary but) some proof theory of $SLK4$ with a stratum number s and formulas of $S4Val$ respectively. Then

- i) there is no tautology in $SLK4$. I.e., there is no formula such that $\vdash_{\text{SLK4}_s} A$ for every $SLK4_s$.
- ii) The law of excluded middle does not hold in $SLK4$. I.e., $\not\vdash_{\text{SLK4}_s} A \vee \neg A$.
- iii) The law of contradiction does not hold in $SLK4$. I.e., $\vdash_{\text{SLK4}_s} A \wedge \neg A$.
- iv) Modus Ponens does not hold in $SLK4$. I.e., $\vdash_{\text{SLK4}_s} A$, and $\vdash_{\text{SLK4}_s} A \supset B \not\Rightarrow \vdash_{\text{SLK4}_s} B$.
- v) Deduction Theorem does not hold in $SLK4$. I.e., $\Gamma, \mathcal{A} \vdash_{\text{SLK4}_s} B \not\Rightarrow \Gamma \vdash_{\text{SLK4}_s} A \supset B$.
- vi) $SLK4$ is paraconsistent. I.e., $A, \neg A \not\vdash_{\text{SLK4}_s} B$.

Proof Obvious. □

4.1 Algorithm

Now, we can proof-theoretically determine the truth value of a formula by the following algorithm:

Algorithm 1

- 0) $s = 0$.
- ia) Prove $\vdash_{\text{SLK4}_s}^d A$. If it succeeds, go to ib). If not, go to ii).
- ib) Prove $\vdash_{\text{SLK4}_s}^d \neg A$. If it succeeds, return B_s as the truth value of A . If not, return T_s as its truth value.
- ii) Prove $\vdash_{\text{SLK4}_s}^d \neg A$. If it succeeds, return F_s as the truth value of A . If not, go to iii).
- iii) If $s < d - 1$, then $s := s + 1$, and go to ia). If not, return N_0 as the truth value of A .

The correctness of Algorithm 1 is secured by the following Theorem 8, together with Theorem 4.

Theorem 8 Let $M_s = \langle D, \pi_s, f, v \rangle$ be an arbitrary model which satisfies the proper axioms and inferential rules of SLK4_s. Then

Algorithm 1 returns the truth value $\tau_s \in \{B_s, T_s, F_s\}$ to $A \Leftrightarrow$ There is the least stratum number s such that $D(\pi_s, f, v, A) = \tau_s$.

Algorithm 1 returns the truth value N_0 to $A \Leftrightarrow$ the value of every $D(\pi_s, f, v, A)$ is N_0 .

Proof

i) Suppose that Algorithm 1 returns the truth value B_s to A . Then by Theorem 6, $D(\pi_s, f, v, A) = B_s$ for the least stratum number s such that $D(\pi_s, f, v, A) = \tau_s$.

Conversely, suppose that there is the least stratum number s such that $D(\pi_s, f, v, A) = B_s$. Then by Theorem 6, $\vdash_{\text{SLK4}_s} A$. So Algorithm 1 returns the truth value B_s to A .

ii,iii,iv) Similarly for the cases where Algorithm 1 returns the truth value T_s, F_s , and N_0 . \square

4.2 Default Reasoning

As stated above, we can realize a certain kind of default reasoning in SLK4. For this, consider the following example. At first, SLK4 has the following proper axiom and inferential rule:

(16) i) $\rightarrow_1 \text{bird}(\text{tweety})$.

ii) $\frac{\Gamma \rightarrow \Delta, \text{bird}(\alpha), \Sigma}{\Gamma \rightarrow \Delta, \text{fly}(\alpha), \Sigma}^1$

From this,

(17) $\vdash_1 \text{fly}(\text{tweety})$.

Now suppose that the following proper axiom and inferential rule are added to the above SLK4:

(18) i) $\rightarrow_0 \text{ostrich}(\text{tweety})$.

ii) $\frac{\Gamma \rightarrow \Delta, \text{ostrich}(\alpha), \Sigma}{\Gamma \rightarrow \Delta, \neg \text{fly}(\alpha), \Sigma}^0$

Then (17) is overridden by

(19) $\vdash_0 \neg \text{fly}(\text{tweety})$,

and a certain kind of non-monotonicity appears.

The proper inferential rules such as (16ii) in the following form ($s \geq 1$)

(20) $\frac{\Gamma \rightarrow \Delta, \alpha, \Sigma}{\Gamma \rightarrow \Delta, \beta, \Sigma}^s$

correspond to the following normal default in its usual form:

(21) $\frac{\alpha : M\beta}{\beta}$

4.3 Truth Maintenance System

SLK4 can also realize a certain kind of truth maintenance system. For this, suppose that SLK4 has the following proper axiom and inferential rule in addition to (16):

(22) i) $\rightarrow_1 \text{dfowl}(\text{tweety})$ (*dfowl*: domestic fowl)

ii) $\frac{\Gamma \rightarrow \Delta, \text{dfowl}(\alpha), \Sigma}{\Gamma \rightarrow \Delta, \neg \text{fly}(\alpha), \Sigma}^1$

Then

(23) i) $\vdash_1 \text{fly}(\text{tweety})$

ii) $\vdash_1 \neg \text{fly}(\text{tweety})$

so that the truth value of $\text{fly}(\text{tweety})$ is B_1 . But then suppose that (18) is added to SLK4. Then (23) is overridden by (19), and the contradiction about Tweety is solved.

Obviously, the default reasoning and the truth maintenance system in 4.2 and 4.3 do not cover all of these reasonings. The former can treat the normal default, but not the non-normal one. And if the 0-th

stratum contains a contradiction A and $\neg A$, the latter cannot solve it. For this, we must use some other TMS method developed so far.

But it is also true that they can realize a certain interesting type of non-monotonic reasoning in a simple, lucid manner without recalculation found in several non-monotonic procedures in [12].

Notice that SLK4 with (16) and (22) derives (23). A usual default theory would derive either (23i) or (23ii), but not both, and it depends on the order of applying the default rules (16ii) and (22ii) which one is derived. In this case, some preferential order among the default rules is generally assumed. But in this paper, such an order is not assumed on a stratum, and the contradiction is accepted – thanks to the 4-valued method – without serious theoretical problems. Instead, the hierarchy among the stratums represents the preferential order here.

Further notice that Algorithm 1 contains the procedure $\not\vdash_s A$. Because of the semi-decidability of the first-order logic, it is in general undecidable. But in SLK4, the proper axioms are restricted to clauses. Further, if the number of proper axioms and proper inferential rules are finite, and the latter contain no circularity, it is possible to construct a decidable system with unification of variables.

5 Conclusion

In this paper, I presented the semantics S4Val of the stratified 4-valued logic and its proof theory SLK4. The former is based on Ginsbergs bilattice DEFAULT, on which SLK4 is developed in a simple, lucid LK-calculus without obscure 4-valued connectives.

SLK4 can treat some types of default reasoning and truth maintenance system. Although they cannot capture all such reasonings, it provides a concise method to treat an interesting part of them. And it will be an useful attempt to integrate it into a Q&A system together with other reasoning tools.

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